Even a tiny positive Λ casts a long shadow

Abhay Ashtekar
Institute for Gravitation and the Cosmos, Penn State

Will first summarize joint work with Béatrice Bonga and Aruna Kesavan in the weak field approximation (CQG+, CQG 32, 025004 (2015); PRD 92, 044011 (2015); PRD 10432 (2015)) and then outline the proposal for gravitational radiation theory full GR AA (in preparation).

Discussions: Bicak, Blanchet, Chrusciel, Costa, Garriga, Goldberg, Lehner, Poisson, & Saulson.

A Century of GR Conference, Berlin. November, 2015

Isolated Systems and Gravitational Waves

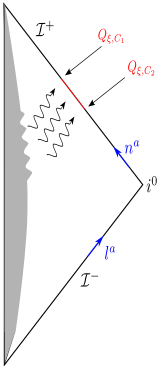
- Confusion regarding the reality of gravitational waves in full GR (Einstein: 1916-18 vs 1936; the Levi-Civita c-metric)

- The Bondi, Penrose et al framework (1960s-1980s):

Notion of null infinity $\mathcal{I}$. Topology $\mathbb{S}^2 \times \mathbb{R}$. Because $\mathcal{I}$ is null, it is ruled by its null normals. This structure reduces the asymptotic symmetry group from Diff($\mathcal{I}$) to the Bondi-Metzner-Sachs Group $\mathcal{B} = \mathcal{S} \rtimes \mathcal{L}$.

- $\mathcal{B}$ admits a unique 4-d normal subgroup $\mathcal{T}$ of translations.

Used critically in the definition of energy-momentum.



- Gravitational radiation: Curvature of the intrinsic connection D on $\mathcal{I}$ defines the Bondi News tensor N_{ab}. No incoming radiation: $N_{ab} = 0$ on $\mathcal{I}^-$. The BMS group naturally reduces to the Poincaré on $\mathcal{I}^-$.

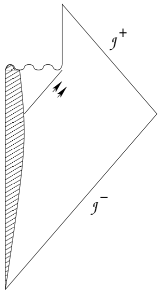
- Bondi 4-momentum and fluxes: Balance laws

$$Q_\xi[C_2] - Q_\xi[C_1] = \int_{\Delta\mathcal{I}^+} \xi |N_{ab}|^2 \, \mathrm{d}^3\mathcal{I}^+$$

Flux is manifestly positive (Bondi: "Gravitational waves are real; you can boil water with them."). Positive energy theorem for $Q_\xi[C]$ (Horowitz & Perry; Schoen & Yau).

Positive Cosmological Constant

• None of the rich structure, just discussed, exists if there is $\Lambda > 0$. We do not have even the basic notions: Bondi news to characterize gravitational radiation; non-trivial balance laws; positive Bondi energy and positive energy-flux; 'no incoming radiation condition' on $\mathcal{I}^-$. Don't know what gravitational waves mean in full, non-linear GR if $\Lambda > 0$, however tiny! (Some of the Difficulties have been pointed out by Penrose, Bičák, Krtouš, Podolský, ... over the years.)



• We do not have a canonical positive and negative frequency decomposition that is needed in the construction of asymptotic Hilbert spaces.

Organization of the talk

1. Asymptotically de Sitter space-times: Unforeseen Difficulties.
2. Linear Theory: Novel features.
3. Generalization of the Bondi-Penrose framework: Proposal
4. Summary and Outlook.

1. $\Lambda > 0$: Unforeseen Difficulties

Gravitational radiation introduces qualitative differences

• Recall first the notion of Asymptotic flatness. A Physical space-time $(\tilde{M}, \tilde{g}_{ab})$ is said to be asymptotically Minkowski if $\tilde{g}_{ab}$ approaches a Minkowski metric as $1/r$ as we recede from sources in null directions. In Bondi coordinates:

$$\mathrm{d\tilde{s}^2 \to -du^2 - 2dudr + r^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)}$$

• Presence of gravitational waves adds an unforeseen twist: there is no longer a canonical Minkowski metric that $\tilde{g}_{ab}$ approaches! The possible Minkowski metrics differ by angle dependent translations (i.e. BMS supertranslations). The asymptotic symmetric group is not the Poincare Group $\mathcal{P} = \mathcal{T} \rtimes \mathcal{L}$ but the BMS group $\mathcal{B} = \mathcal{S} \rtimes \mathcal{L}$. The BMS group $\mathcal{B}$ reduces to $\mathcal{P}$ if there is no radiation, i.e. for the class of space-times with $N_{ab} = 0$.

• It is not even larger, i.e., Diff($\mathcal{I}^+$), because there is extra structure: $\mathbb{S}^2 \times \mathbb{R}$ topology and, more importantly, the ruling of $\mathcal{I}^+$ by its null normal.

The first key difficulty in a nutshell

• One would like to say that a Physical space-time $(\tilde{M}, \tilde{g}_{ab})$ is asymptotically de Sitter if $\tilde{g}_{ab}$ approaches a de Sitter metric as $1/r$ as we recede from sources in null directions. This condition is indeed satisfied by stationary space-times such as Kerr-de Sitter. Then the asymptotic symmetry group is just the de Sitter group $\mathcal{D}$. Allows us to define de Sitter momenta (mass, angular momentum, ...).

• But in presence of gravitational waves, there is an entirely new twist: Now $\tilde{g}_{ab}$ deviates from de Sitter metric (in a controlled fashion) even to leading order! For example, in the axi-symmetric case, an appropriate generalization of the Bondi ansatz gives, to leading order, (He, Cao)

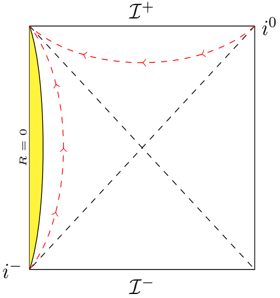
$$\mathrm{ds^2 \to -(1-(\Lambda/3)r^2)du^2 - 2dudr + r^2\,(e^{2\Lambda f}\,d\theta^2 + e^{-2\Lambda f}\,\sin^2\theta\,d\varphi^2)}$$

where $f = f(u, \theta, \phi)$. If $\dot{f} = 0$ no radiation radiation at $\mathcal{I}^+$ (AA, Bonga, Kesavan). But by inspection, de sitter metric results at infinity only if $f =$ const.

• In presence of gravitational waves, then, the asymptotic symmetry group is not the de Sitter group $\mathcal{D}$ but Diff($\mathcal{I}^+$). No semi-direct product structure; No notion of 'de Sitter momentum or angular momentum.'

More precisely ...

- A physical space time $(\tilde{M},\, \tilde{g}_{ab})$ is said to be asymptotically de Sitter if it admits a conformal completion $(M,\, g_{ab})$, where $M = \tilde{M} \cup \mathcal{I}$ is a manifold with boundary $\mathcal{I}$ and $g_{ab} = \Omega^2 \tilde{g}_{ab}$ s.t.
(i) At the boundary $\mathcal{I}$, we have $\Omega = 0$ and $\nabla_a \Omega \neq 0$;
(ii) $\tilde{g}_{ab}$ satisfies Einstein's equations $\tilde{G}_{ab} + \Lambda \tilde{g}_{ab} = 8\pi G_{\rm N} \tilde{T}_{ab}$, with $\tilde{T}_{ab}$ falling off appropriately; and,
(iii) $\mathcal{I}$ is topologically $\mathbb{S}^3$ (minus punctures, e.g. $\mathbb{S}^2 \times \mathbb{R}$) and complete in an appropriate sense.



- Field equations now imply that $\mathcal{I}$ is space-like so its normal is no longer tangential to it. Hence now $\mathcal{I}$ does not have an extra structure like a preferred ruling. Asymptotic symmetry group is just Diff$(\mathcal{I})$! Not clear how to define energy, momentum, or angular momentum at $\mathcal{I}$.

Strengthening the boundary conditions removes gravitational waves!

• Can we strengthen the boundary conditions to reduce Diff($\mathcal{I}$) to a manageable size? A natural strategy, commonly used in the literature is to demand that the intrinsic metric on $\mathcal{I}$ be conformally flat. Natural because the intrinsic geometry of $\mathcal{I}$ is then the same as in de Sitter space.

• Not only is the group reduced but it is reduced precisely to the de Sitter group! One can define Bondi-like charges $Q_\xi[C] = \oint_C E_{ab}\xi^a \mathrm{dS}^{\mathrm{b}}$. Yield expected answers in Kerr-de Sitter.

• However, the condition is too strong! (Friedrich) Conformal flatness of intrinsic geometry $\Leftrightarrow$ $B_{ab} = 0$ at $\mathcal{I}$. Since $\mathcal{I}$ is space-like, half the solutions simply thrown out. Physical restriction: In cosmological perturbations, for example, it removes by hand the 'growing mode', leaving behind only the 'decaying mode'! Furthermore, all de Sitter fluxes associated with these remaining solutions vanish identically! In the full theory, $Q_\xi[C]$ are well-defined but absolutely conserved! No flux of de Sitter energy, momentum or angular momentum!

• Contrast with the AdS case: Since $\mathcal{I}$ is time-like there, an additional 'reflective' boundary condition is needed to make the evolution well-defined. So absolute conservation of $Q_\xi[C]$ is physically reasonable e.g. in the AdS/CFT analysis.

2. Linear fields on de Sitter

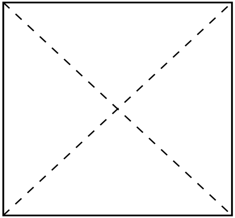
• We have a quandary in full non-linear GR. Practical Strategy: Bypass it by going to the weak field limit; analyze key issues; and return to the full theory using guidance from the linear analysis. There are surprises also in the linear theory!

• The background de Sitter space-time provides isometries. Straightforward to define the corresponding de Sitter momenta for test fields, say, Maxwell: $F_\xi = \int_\Sigma \tilde{T}_{ab}\, \xi^a \mathrm{dS^b}$; can take limit $\Sigma \to \mathcal{I}^+$. But for gravitational waves, we do not have a stress-energy tensor. Can use symplectic methods instead.

• Covariant phase space $\Gamma_{\rm cov}$ consists of space of regular solutions to Maxwell's equations and is equipped with a symplectic structure:

$$\omega(A, A') = \int_\Sigma (A_a F'^{ab} - A'_a F^{ab})\, \mathrm{dS_b};$$

which is conserved and gauge invariant.



• Infinitesimal transformation $A_a \to A_a + \epsilon \mathcal{L}_\xi A_a$ preserves ω and the Hamiltonian $\mathbf{H}_\xi := \frac{1}{2}\omega(\mathcal{L}_\xi A, A)$ exactly equals F_ξ!

• These Hamiltonian methods can be applied to the linearized (and indeed, full) GR, bypassing the need of a stress-energy tensor.

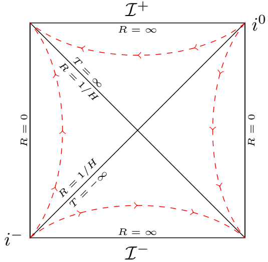
de Sitter momentum fluxes

- To compute energy, momentum and angular momentum carried by gravitational waves, start with the covariant phase space $\Gamma_{\rm cov}$ of linearized solutions (on a Poincaré patch –with an eye to the quadrupole formula). Then,

$$\omega(h,h') = \frac{1}{H\kappa}\int_\Sigma h_{ab}E'^{ab} - h'_{ab}E^{ab})\mathrm{dV}, \quad \text{where } H = \sqrt{\Lambda/3}.$$

- We can calculate Hamiltonians $\mathbf{H}_\xi = \frac{1}{2}\omega(\mathcal{L}_\xi h, h)$ correspond to any de Sitter symmetry ξ^a. A de Sitter 'time translation' T^a yields de sitter 'energy':
$\mathbf{H}_T = \frac{1}{2H\kappa}\int_{\mathcal{I}^+} E^{ab}\,(\mathcal{L}_T h_{ab} - 2H\,h_{ab})\,\mathrm{dV}$

  - Note that gravitational waves can carry arbitrarily large negative de Sitter energy, no matter how tiny Λ is. The limit $\Lambda \to 0$ is subtle but well-defined and we recover the standard positive definite answer in Minkowski space-time. Thus, the lower bound of energy carried by gravitational waves has an infinite discontinuity! Same holds for electromagnetic waves in de Sitter.



- Note also that if B_{ab} vanishes on $\mathcal{I}^+$, so does h_{ab}. Hence these gravitational waves in de Sitter carry no energy (or momentum and angular momentum).

The Quadrupole formula

- During 1916-18, Einstein used the first post-Minkowskian, first PN approximation, to obtain the celebrated quadrupole formula:

$$\dot{E} = \frac{G}{8\pi} \int_{\mathcal{I}^+} (\dddot{Q}_{ab} \dddot{Q}^{ab}_{\rm TT})_{\rm Ret} \;\; d\mathcal{I}^+$$

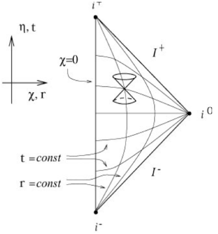
The problem of extending it to the $\Lambda > 0$ case has been open for almost a century because a host of unforeseen difficulties arise no matter how tiny Λ is!

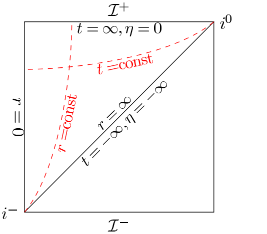
(i) Gravitational waves can carry arbitrarily large negative energy. Potential for instability! Physical quantities can be discontinuous in the $\Lambda \to 0$ limit.

(ii) For $\Lambda = 0$ one considers energy fluxes across time-like cylinders r=const approaching $\mathcal{I}^+$, and makes a heavy use of the $1/r$-expansions. But in de Sitter space-time, these cylinders approach a past cosmological horizon (across which there is no energy-flux for retarded solutions) rather than $\mathcal{I}^+$. The familiar $1/r$-expansions no longer useful!

(iii) A tail term in the retarded solution already in the first post-de Sitter order. At $\mathcal{I}^+$, as significant as the sharp term.

(iv) wave-lengths increase as the wave propagates, making the geometrical optics approximation invalid near $\mathcal{I}^+$.





Generalization to include $\Lambda > 0$

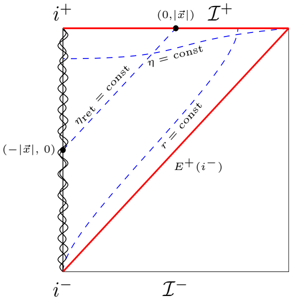
• Find retarded solution in the first post de Sitter, first PN approximation and then energy flux using Hamiltonian methods. The final expression has the form:

$$\dot{E}_T = \frac{G}{8\pi} \int_{\mathcal{I}^+} (\mathcal{R}_{ab}\, \mathcal{R}^{ab}_{\rm TT})\, \mathrm{d}\mathcal{I}^+ \quad \text{where,}$$

$$\mathcal{R}_{ab} = \left[\dddot{Q}^{(\rho)}_{ab} + 3H\ddot{Q}^{(\rho)}_{ab} + 2H^2\dot{Q}^{(\rho)}_{ab} + H\ddot{Q}^{(p)}_{ab} + 3H^2\dot{Q}^{(p)}_{ab} + 2H^3 Q^{(p)}_{ab}\right]_{\rm Ret}$$

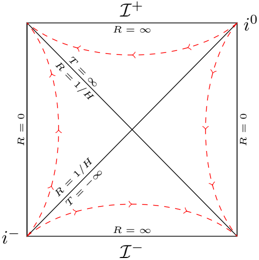
• We know from the Raychaudhuri equation in cosmology that pressure contributes to gravitational attraction. We now learn that it also sources gravitational radiation! Lower derivative terms also for the standard (density) quadrupole.

• One can show that the energy radiated is positive definite: Although a neighborhood of $\mathcal{I}^+$ does admit gravitational waves with negative energy of arbitrarily large magnitude, they cannot be produced by a time changing quadrupole!

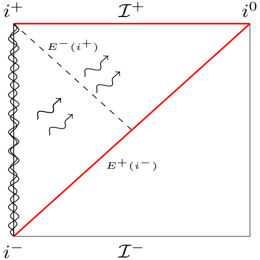


• In the limit $\Lambda \to 0$ we recover the Einstein formula. Furthermore, there is full control to systematically calculate the corrections due to non-zero Λ.

What happened to the $\Lambda > 0$ Difficulties?



(i) Positivity of energy:
energy reaching $\mathcal{I}^+$ can be negative only because T^a is past-directed along a portion of the cosmological horizon $E^+(i^-)$. But retarded solutions have no flux across $E^+(i^-)$. So the energy-flux through $\mathcal{I}^+$ is positive!



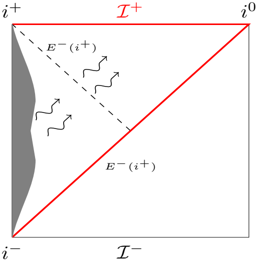
(ii) The $1/r$ expansion is indeed not useful. But one can replace it with a new late time expansion near $\mathcal{I}^+$.

(iii) The retarded solution has a non-trivial tail term. But what matters for energy loss are time derivatives and their propagation is sharp. But the tail term in the solution essential to make the flux well-defined.

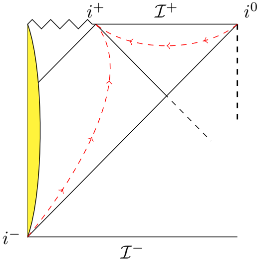
(iv) Since time derivatives $\mathcal{R}_{ab}$ in the in the energy loss formula is evaluated at the retarded time instant, what enters is the wave length at the source, not in the asymptotic region.

3. Full non-linear GR: Proposal

• As in the development of the $\Lambda = 0$ theory, let us use insights from the linear theory to develop the $\Lambda > 0$ analog of the Bondi-Penrose framework.



• Isolated Systems that remain spatially bounded define points $i^\pm$ on $\mathcal{I}^\pm$. System and radiation it emits is visible only to the future of the cosmological horizon $E^+(i^-)$. So we focus only on this region.

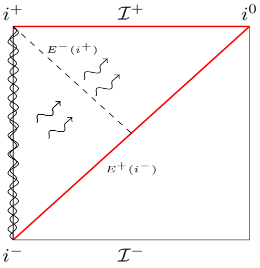


• In the quadrupole formula, the 'no incoming radiation condition' imposed across $E^+(i^-)$. Plays a key role in assuring positivity of energy flux at $\mathcal{I}^+$. So we ask that $H^- := E^+(i^-)$ be a weakly isolated horozn: Topology $\mathbb{S}^2 \times \mathbb{R}$; non-expanding null surface whose null normal ℓ^a is a symmetry of the intrinsic metric and the 'extrinsic curvature' of H^-. (AA, Beetle, Lewandoswki,...). H^- replaces $\mathcal{I}^-$ of Asymptotically Minkowski space-times.

Symmetries and Bondi-type Charges

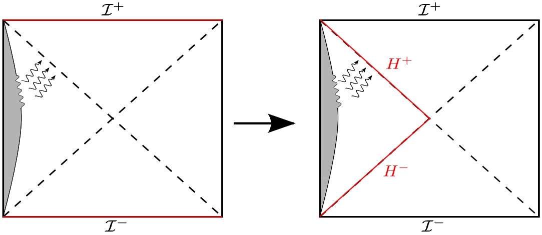
• Conditions on H^- satisfied in Kerr-deSitter, and numerical simulations of collapse. (Shibata and Shapiro groups)

• If H^- is axi-symmetric, it carries a 7-dimensional symmetry group. Can define energy, momentum and angular momentum using the electric part of the Weyl tensor, E^{ab}, on H^-. Absolutely conserved as expected and standard answers in Kerr-de Sitter. Thus good control on the past boundary that replaces $\mathcal{I}^-$.



• Two strategies for analyzing radiation at future infinity are being pursued. In the first, One works with $\mathcal{I}^+$ without imposing the condition $B_{ab} = 0$ at $\mathcal{I}^+$. So the intrinsic +,+,+ metric q_{ab} at $\mathcal{I}^+$ is not conformally flat. The idea is to extract a fiducial equivalence class of conformally flat metric $\{\mathring{q}_{ab}\}$ and use the de Sitter group it selects. Only partial results so far.

A Second Strategy: 'Local' $\mathcal{I}^+$



• Local $\mathcal{I}^+$: Restricting to the 'local universe' of 5 GPc size around us. Idea: Use the past cosmological horizon $H^+ := E^-(i^+)$ in place of $\mathcal{I}^+$. Note the resemblance to the conformal diagram of Asymptotically Minkowski space-times! $H^\pm$: local $\mathcal{I}^\pm$.

• Using the structure at the bifurcate horizon, one can drag the Weakly isolated horizon structure from H^- to H^+. The symmetries of this fiducial WIH enable one to define Bondi-like charges and fluxes across H^+. For example, Bondi-type energy

$$Q_T[C] = \frac{1}{\kappa} \oint_C r\big[\mathrm{Re}(\Psi_2 + \bar\sigma_{(\ell)}\sigma_{(\mathrm{n})}) + \theta_{(\mathrm{n})}\,(\tfrac{1}{\mathrm{r}} - \tfrac{1}{2}\,\theta_\ell)\big]\mathrm{d}^2\mathrm{V}$$

$$= \frac{r}{2G}\big[1 - H^2 r^2 + 2\dot r\big]$$ related to the area of C! (as expected of horizons).

$$\to \frac{1}{\kappa} \oint_C r\big[\mathrm{Re}(\Psi_2 + \bar\sigma_{(\ell)}\sigma_{(\mathrm{n})})\big]$$ in the $\Lambda \to 0$ limit.

• $Q_T[C]$ would be positive if $r < 1/H$ the cosmological radius.

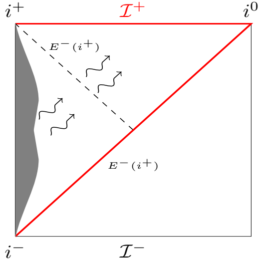
4. Summary and Outlook

• Primary motivation: conceptual. Some 100 Years have passed since Einstein's discovery quadrupole formula and some 50 years since the Bondi-Penrose framework. For 15 years, we have known that the accelerated expansion of the universe is best explained by a positive Λ. Now, numerical relativists, observers and experimentalists have taken us to the dawn of the new era of gravitational wave science. So it is high time that we have a firm theoretical framework describing gravitational waves in GR with $\Lambda > 0$. (Recall the confusion about reality of gravitational waves during the first 50 years of GR!)

• The issue of this extension has been open so long because inclusion of Λ, however small, introduces novel conceptual issues both in full theory and in the linear approximation. These arise because the asymptotic space-time structure changes non-trivially: $\mathcal{I}^+$ is space-like rather than null. Hence problems persist if Λ were to be replaced by some other form of 'dark energy' so long as the accelerated expansion continues to the future.

• Stability of $\mathcal{I}^+$ for $\Lambda > 0$ was established in a pioneering work by Friedrich in 1991. But the problem of extracting physical information has been open: Bondi news; energy, momentum and angular momentum 2-sphere integrals; expressions of fluxes of these quantities; relation between the radiated power to properties of sources in the weak field, slow motion limit, ... Even a tiny Λ casts a long shadow!

• These issues have now been resolved in the weak field limit: Post de Sitter, first post-Newtonian approximation. A priori it is not obvious that tiny Λ can only make negligible corrections because the limit is discontinuous in important ways: $\mathcal{I}^+$ changes its character. But detailed analysis provides systematic ways of calculating the 'error' terms and shows why and how the concerns can be by-passed. For full, non-linear GR, well-developed strategies but further work is needed.



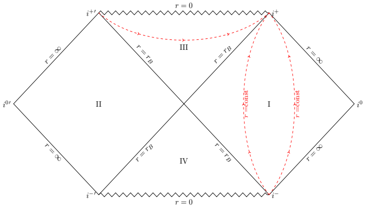
• Open issues: Examples:

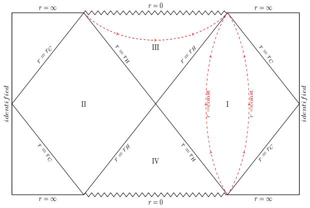
(i) Is the analog of Bondi-energy 2-sphere integral positive if the matter satisfies energy conditions and H^- is a weakly isolated horizon? Recall the importance of the positive energy theorem in geometric analysis.

(ii) Is the radiated flux positive (since there is no energy flux across H^-) as in the new quadrupole formula? If not, there would be gravitational instabilities.

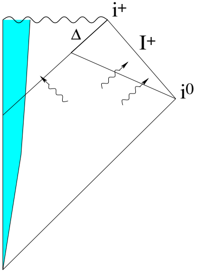
Comment: Definitions of de Sitter momenta of Abbott & Deser; Kelley & Marolf; Chruściel, Jezierski & Kijowski; ... refer to i^o. Positive energy theorems of Kastor & Traschen; Luo, Xie and Zhang also refer to i^o and, furthermore, a conformal Killing field in de Sitter, which is not an asymptotic symmetry. Szabodas & Tod: Positive charge but interpretation unclear.

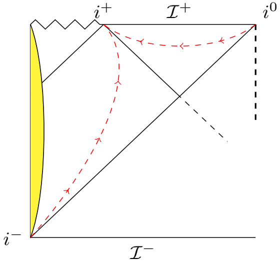
Gravitational Collapse & BH evaporation





$\Lambda = 0$ versus $\Lambda > 0$ gravitational Collapse





Conceptual problems in specifying asymptotic Hilbert spaces for the Hawking radiation resolved in the new scenario: incoming states can be specified on H^- and outgoing on H^+ , even allowing for back-reaction due to outgoing radiation.