ARE BLACK HOLES REAL

?

A mathematical response

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TWO NOTIONS OF REALITY

MATHEMATICAL REALITY

An object is **real** if it is mathematically consistent.

PHYSICAL REALITY

A mathematical model is **real** if it leads to effects verifiable by experiments.

Can physical reality be tested by mathematical means, in the framework of a given theory?

EXAMPLE:

Black holes are specific solutions of the Einstein field equations.
They exist as real, rich and beautiful mathematical objects, which deserve to be studied for their own sake.
They are also consistent with many indirect astrophysical observations.

BUT ARE THEY REAL ?

Stationary, asymptotically flat, solutions of the Einstein field equations (in vacuum),

$$\mathrm{Ric}(g) = 0$$

DEFINITION [External Black Hole]

- Asymptotically flat, globally hyperbolic, Lorentzian manifold with boundary **(M, g)**, diffeomorphic to the complement of a cylinder in $\mathbf{R}^{1+3}$.
- Metric **g** has an **asymptotically timelike, Killing** vectorfield **T**,

$$\mathbf{L_T\, g = 0}:$$

- Completeness (of Null Infinity)

KERR FAMILY $K(a, m),\ 0 \le a \le m$

Boyer-Lindquist (t, r, θ, φ) coordinates

$$-\frac{\rho^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{\rho^2}\left(d\phi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2(d\theta)^2,$$

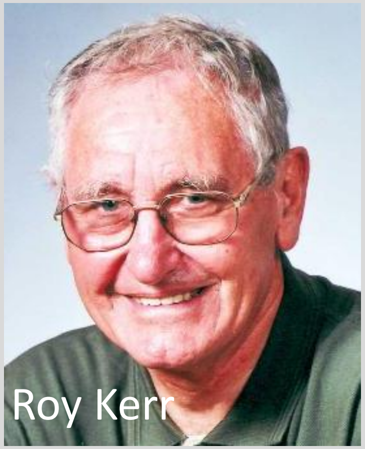
$$\begin{cases} \Delta = r^2 + a^2 - 2mr; \\ \rho^2 = r^2 + a^2(\cos\theta)^2; \\ \Sigma^2 = (r^2 + a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases}$$

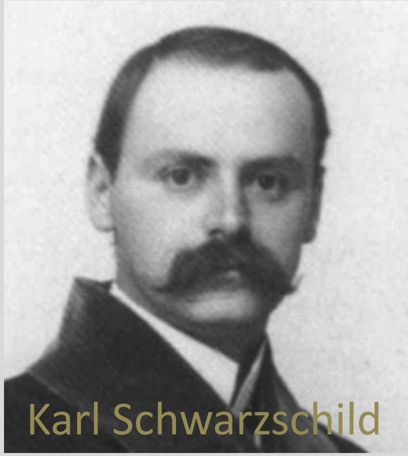
Stationary

$T = \partial t$

Axisymmetric

$Z = \partial\phi$





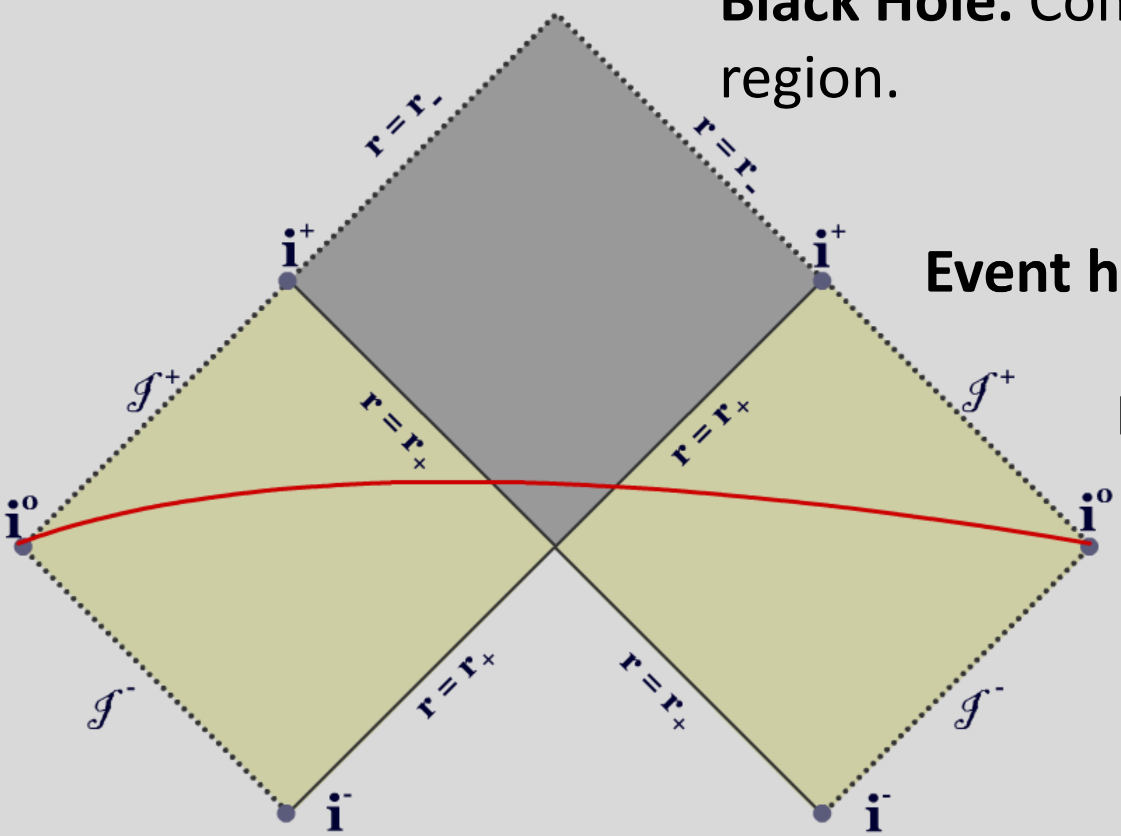
Schwarzschild $a = 0,\ m > 0,$

static, spherically symmetric.

$$-\frac{\Delta}{r^2}(dt)^2 + \frac{r^2}{\Delta}(dr)^2 + r^2 d\sigma_{\mathbb{S}^2}, \qquad \frac{\Delta}{r^2} = 1 - \frac{2m}{r}$$

Maximal Extension

$$\Delta(r_-) = \Delta(r_+) = 0 \quad \Delta = r^2 + a^2 - 2mr$$

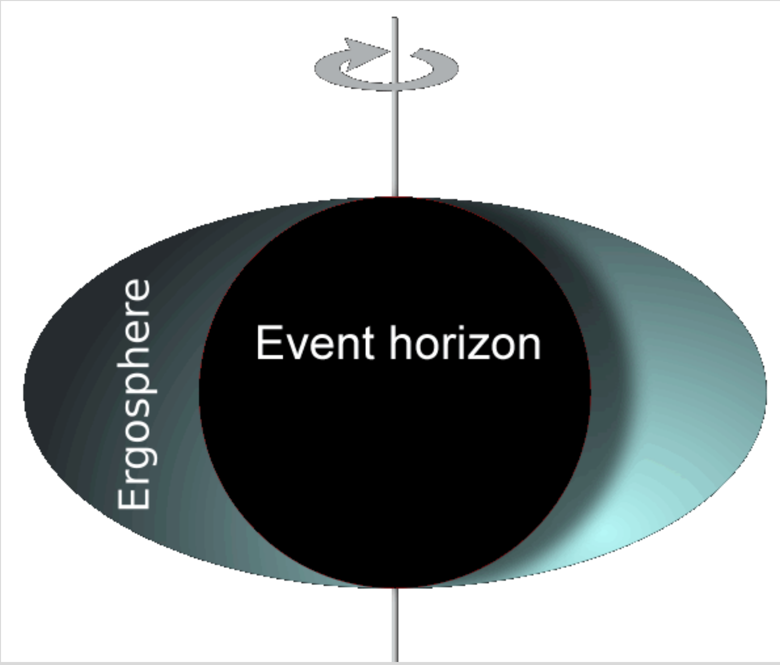


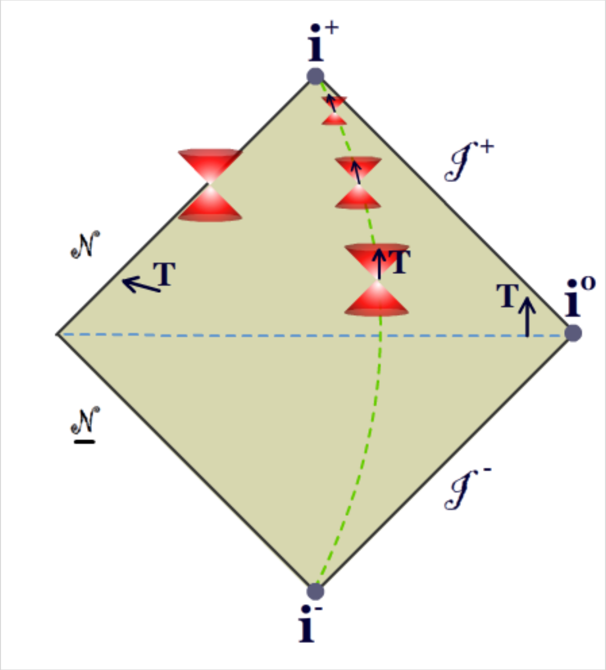
Black Hole. Complement of the external region. $r < r_+$

Event horizon. $r = r_+$

External region. $r > r_+$

EXTERNAL KERR





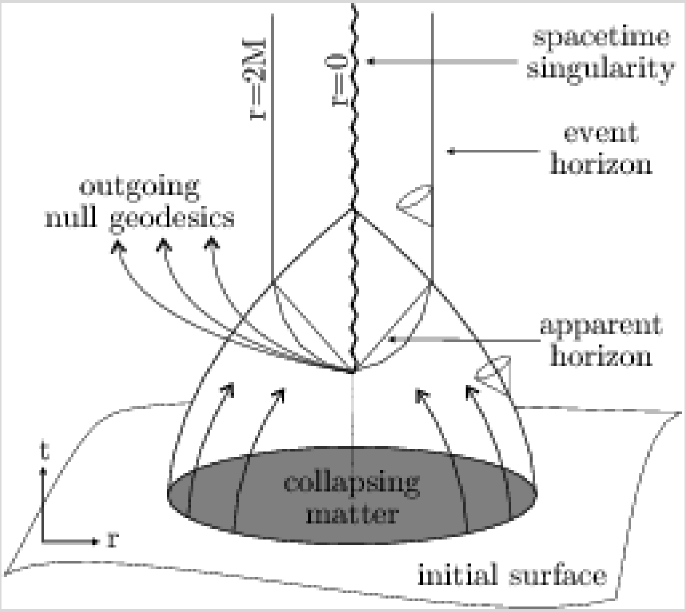
Stationary, axisymmetric

Nontrivial ergoregion. Non-positive energy

Region of trapped null geodesic

STANDARD PICTURE

Large concentrations of matter (energy, curvature) may lead to the formation of a **dynamical** black hole settling down, by gravitational radiation, to a Kerr or Kerr-Newman stationary black hole.



PRESUPPOSES:

- Large concentrations of mass (energy, curvature), lead to the strong causal deformations of Black-Holes!
- All stationary states are Kerr, or Kerr-Newman, black holes.
- These latter are stable under general perturbations.

TESTS OF REALITY $\mathrm{Ric}(g) = 0$

RIGIDITY Does the Kerr family $K(a, m)$, $0 \leq a \leq m$, exhaust all possible vacuum black holes ?

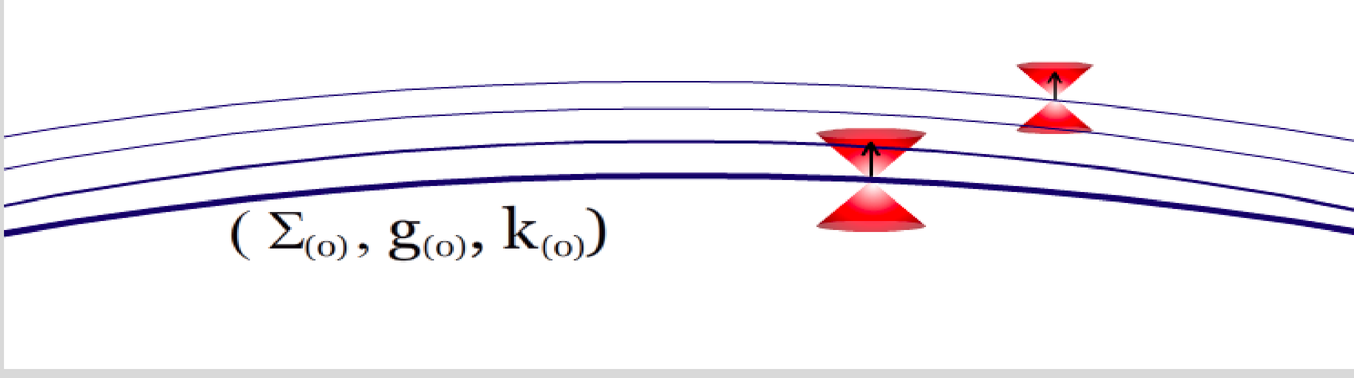
STABILITY Is the Kerr family stable under arbitrary small perturbations ?

COLLAPSE Can black holes form starting from reasonable initial data configurations ?

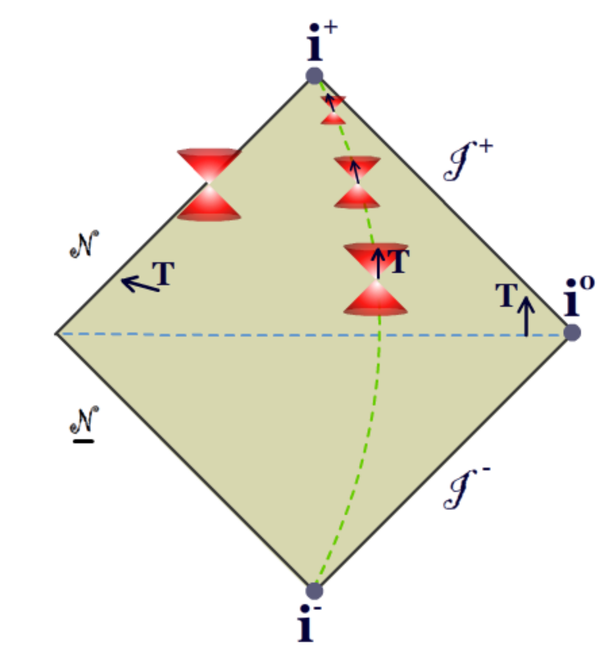
INITIAL VALUE PROBLEM:

Specify initial conditions on a given initial hypersurface and study its maximal future, globally hyperbolic development. J. Leray, Y. C. Bruhat (1952)

$$\mathrm{Ric}(g) = 0$$



TESTS OF REALITY: RIGIDITY



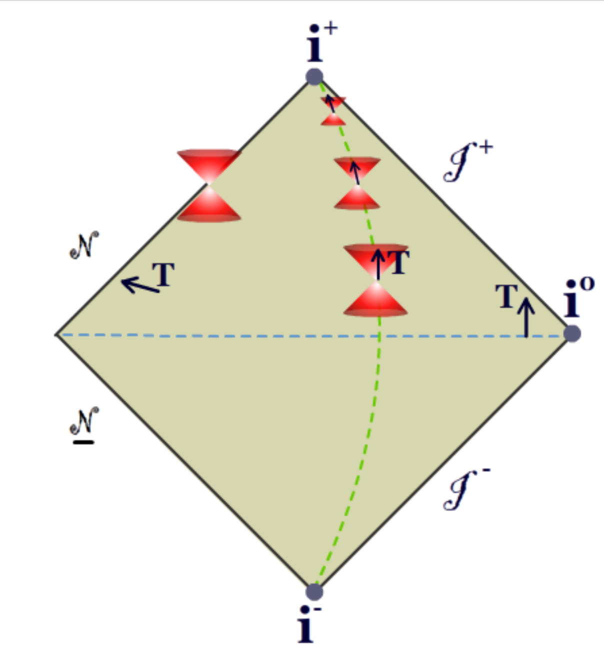
RIGIDITY CONJECTURE

Kerr family K(a, m), $0 \leq a \leq m$ exhaust all stationary, asymptotically flat, regular vacuum black holes.

Despite common perceptions the conjecture has not been settled!

- True in the axially symmetric case. Carter-Robinson
- True in general, under an analyticity assumption. Hawking
- True close to a Kerr space-time. Alexakis-Ionescu-Kl

TESTS OF REALITY: RIGIDITY: new approach



Hawking:

- There exists a second Killing v-field H along the horizon.
- Extending H leads to an ill posed problem.

New approach:

- **Design:** a unique continuation argument to extend H.
- **Obstruction:** the possible presence of T-trapped null geodesics

No such objects in Kerr, or close to Kerr!

CONCLUSIONS

- There exist no other explicit stationary solutions.
- There exist no other stationary solutions close to Kerr, (Kerr-Newman).
- Arguments based purely on the continuation of the Hawking v-field H from the horizon are bound to fail.
- The full problem is far from being solved. Surprises?

CONJECTURE Alexakis-Ionescu-Kl

Rigidity conjecture holds true provided there are no **T-trapped** null geodesics.

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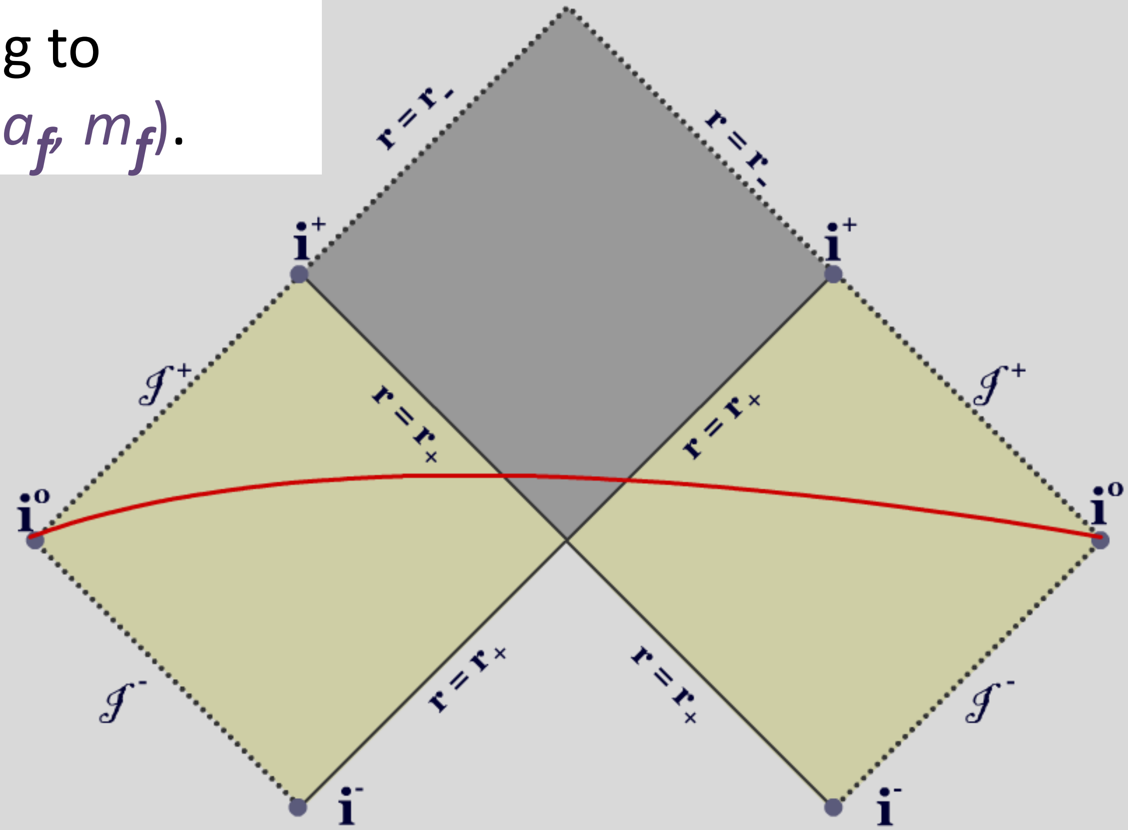
CONJECTURE Stability of (external) Kerr

Small perturbations of a given exterior Kerr ($K(a, m)$, $0 \le a \le m$) initial conditions have max. future developments converging to **another** Kerr solution $K(a_f, m_f)$.



TESTS OF REALITY: STABILITY

The treatment of perturbations of Kerr spacetime has been prolixious in its complexity. Perhaps at a later time, the complexity will be unravelled by deeper insights. But meantime the analysis has led into a realm of the rococo, splendorous, joyful and immensely ornate.

S. Chandrasekhar

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According to the common perception (in Physics!) the stability problem has been solved **by separation of variable methods.**

- **Schwarzschild**. Regge-Wheeler(1957), Vishvevshara(1970), Zerilli(1970)
- **Kerr**. Teukolski, Press- Teukolski(1973)

Whiting(1989) Various linear equation in Kerr, including the Teukolski linearized gravity equations, have **no exponentially growing** modes.

STABILITY: FAR FROM BEING SETTLED!

If lack of exponential growing modes for the linearized equations were enough to deduce nonlinear stabilty, the presence of shock waves, extreme sensitivity to data and turbulence in fluids would be ruled out!

- Lack of exponentially growing modes is necessary but far from sufficient to establish boundedness of solutions to the linearized equations. Example: recent results on the instability of AdS black holes.

- One needs not only boundedness but also sufficiently strong time decay estimates to make sure that the nonlinear term remain negligible through the entire evolution.

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If lack of exponential growing modes for the linearized equations were enough to deduce nonlinear stabilty, the presence of shock waves, extreme sensitivity to data and turbulence in fluids would be ruled out!

- Precise quantitative decay estimates are still insufficient to control the nonlinear terms. Precise structure of the quadratic terms is essential.
- Stability of the Minkowski space is trivial at linearized level and yet it has required a wealth of mathematical ideas and over 500 pages to settle.
- In fact weak type* of linear instabilities are to be expected in view of the fact that the final Kerr solution differs from the one we perturb.

* Leading to lack of decay for the linearized fields

TESTS OF REALITY: STABILITY

MAIN DIFFICULTIES:

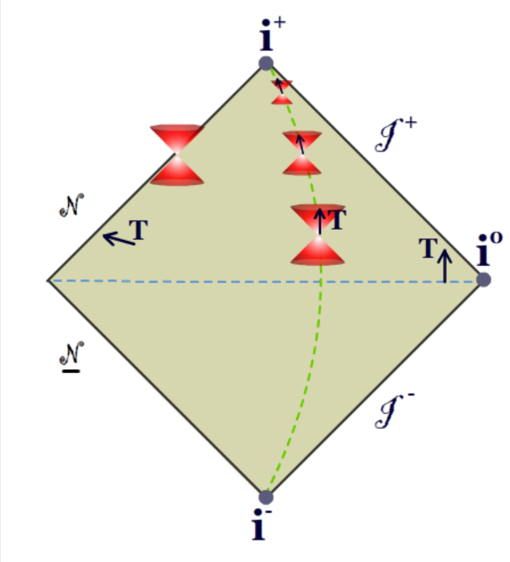
- Until recently even the simplest linear wave equations on fixed black holes backgrounds were not understood.

- Linearized gravity system (LGS), as discussed by Teukolski and al., is not conservative. Thus one cannot establish, even formally, the boundedness and decay of solutions.

- LGS must in fact have real instabilities corresponding to mass, angular momentum and infinitesimal Lorentz transformations.

TESTS OF REALITY: STABILITY far from settled

THEOREM The scalar **wave equation** $\Box_g = 0$ **is strongly stable** on all Kerr backgrounds $K(a, m), 0 \leq a < m$



- Degeneracy of the horizon
- Ergoregion. Non-positive Energy!
- Trapped null geodesics
- Low decay at null infinity

OLD VECTORFIELD METHOD Flexible geometric method to derive quantitative decay based on the symmetries of Minkowski space.

NEW VECTORFIELD METHOD Uses, in addition, locally defined vectorfields to deal with the black hole degeneracies.

TESTS OF REALITY: STABILITY main results

- Global Stability of Minkowski
  Christodoulou-Kl (1990)
- Mode stability of the Kerr family
  Whiting (1989)
- Stability for scalar linear waves (K(a, m), $0 \le a < m$).
  Dafermos -Rodnianski - Shlapentokh Rothman (2014)
- Stability of LGS near Schwarzschild.
  Dafermos - Holzegel - Rodnianski (?)

CONJECTURE
Partial stability

The stability conjecture is true, at least for small, axially symmetric, perturbations of a given Kerr K(a, m)

WORK IN PROGRESS

Two model problems connected to the partial stability conjecture:

1. Stability of Schwarzschild with respect to axially symetric, polarized perturbations.
2. **Half-Linear** stabilty of axially symmetric, perturbations of Kerr Ionescu-Kl (2014).

CONJECTURE

The Kerr family is stable under small axially symmetric **polarized** perturbations.

FACT Consider spacetime **(M, g)** posessing a hypersurface orthogonal, circular, Killing v-field **Z**. Then,

$$\mathbf{g} = g + X d\varphi^2, \qquad X = \mathbf{g}(\mathbf{Z}, \mathbf{Z})$$

and the vacuum Einstein equations becomes

$$\begin{cases} \Box_g X & = \frac{1}{X} D_a X D^a X \\ \mathrm{Ric}(g)_{ab} & = \frac{1}{2X} D_a D_b X - \frac{1}{4X^2} D_a X D_b X \end{cases}$$

CONJECTURE

The Kerr family is stable under small axially symmetric **polarized** perturbations.

EVIDENCE

- There exist a gauge invariant scalar q, at the level of four! covariant derivatives of **X**, which verifies a **good** wave eq.,

$$\square_{\mathbf{g}} \mathrm{q} + V \mathrm{q} = O(\epsilon^2), \qquad V = \frac{8m}{r^3} - \frac{4}{r^2 \sin^2\theta}$$

- Can derive decay estimates for solutions of $\square_{\mathbf{g}} \mathrm{q} + V \mathrm{q} = 0$ in a similar manner as for solutions to

$$\square_{\mathrm{Schw}} \Phi = 0$$

TESTS OF REALITY: STABILITY of SCHWARZSCHILD

EVIDENCE

- The full curvature tensor **R** can be reconstructed from **q** and the linearized gravity system (LGS). This highly nontrivial step requires one to understand the weak nonlinear instabilities of LGS.
- Once the weak instabilities have been accounted decay of solutions to LGS can be derived as in the announced work of **Dafermos-Holzegel-Rodnianski**.
- The remaining $O\,(\cancel{C}^2)$ terms may be treated as in the stability of Minkowski ?.

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GOAL

Investigate the mechanism of formation of black holes starting with reasonable initial data configurations.

TRAPPED SURFACE

Concept introduced by Penrose in connection to his incompleteness theorem.

FACT

Together with Weak Cosmic Censorship conjecture, the incompleteness theorem implies that the presence of a trapped surfaces detects the presence of a black hole.

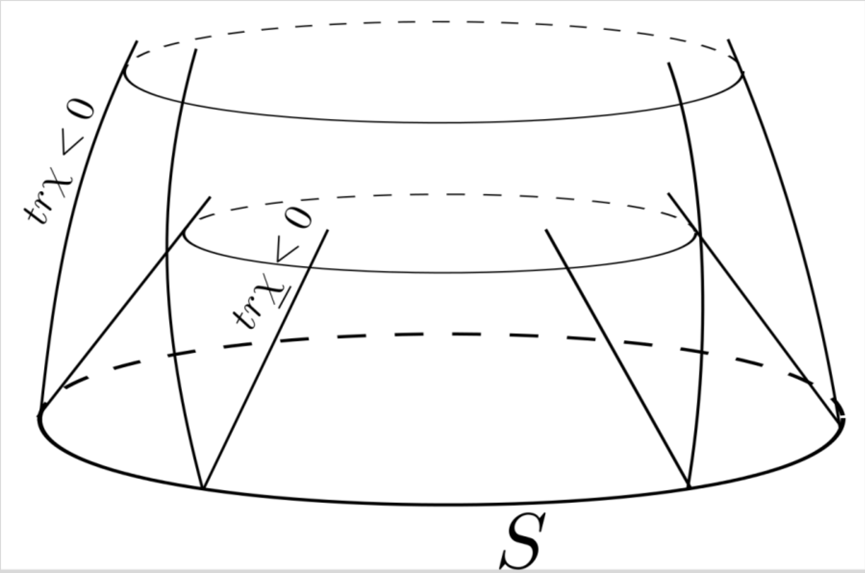
TESTS OF REALITY: COLLAPSE: Penrose

THEOREM

Space-time ***(M; g)*** cannot be future null geodesicaly complete, if

- Ric($\boldsymbol{g}$)(**L**; **L**) ≥ 0;   **L** ***null***
- ***M*** contains a non-compact Cauchy hypersurface
- ***M*** contains a closed **trapped** surface ***S***

Null expansions trX , $tr\underline{X}$

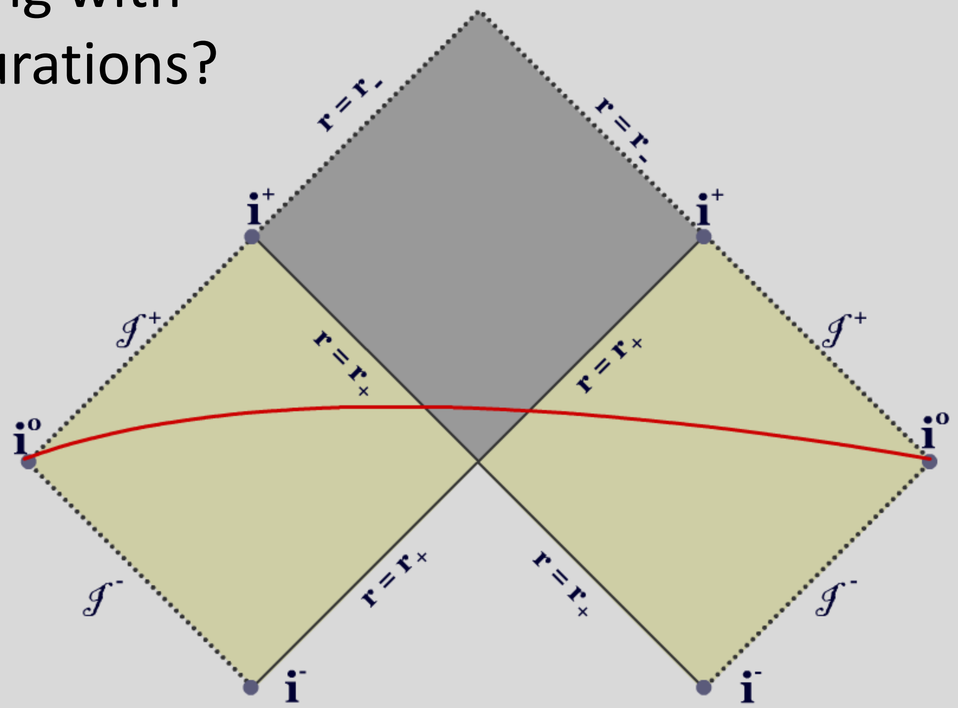


QUESTIONS

- Can trapped surfaces form in evolution ? In vacuum ?
- Does the existence of a trapped surface implies the presence of a Black Hole ?

  True if **weak cosmic censorship** holds true.

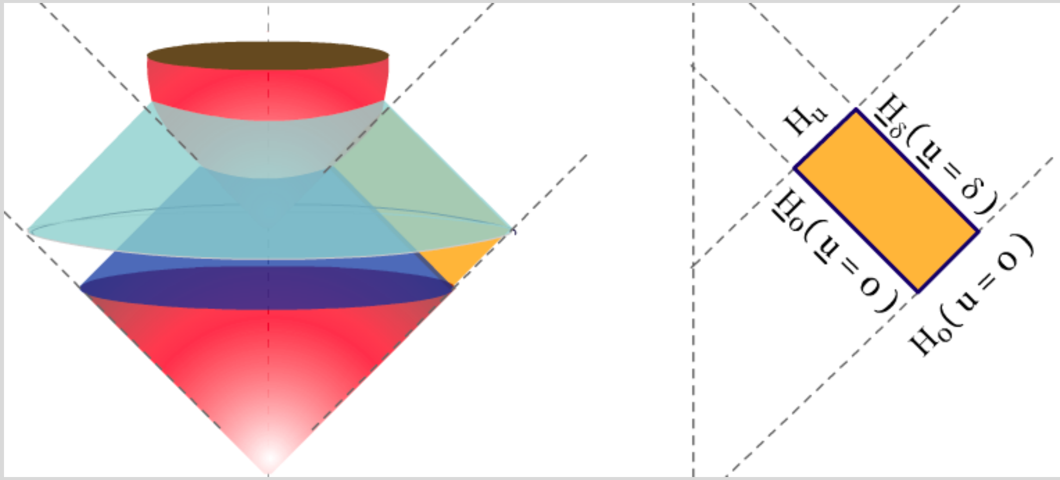
- Can singularities form starting with **non-isotropic**, initial configurations?



MAIN RESULTS

THEOREM
Christ (2008)

There exist an open set of regular, vacuum, data whose MGFHD contains a trapped surface.



1. Specify **short pulse** characteristic data, for which one can prove a general semi-global result of a **double null** foliated spacetime, with **detailed control.**
2. If, **in addition**, the data is sufficiently large, **uniformly** along all its null geodesic generators, a trapped surface must form.

FORMATION OF TRAPPED SURFACES

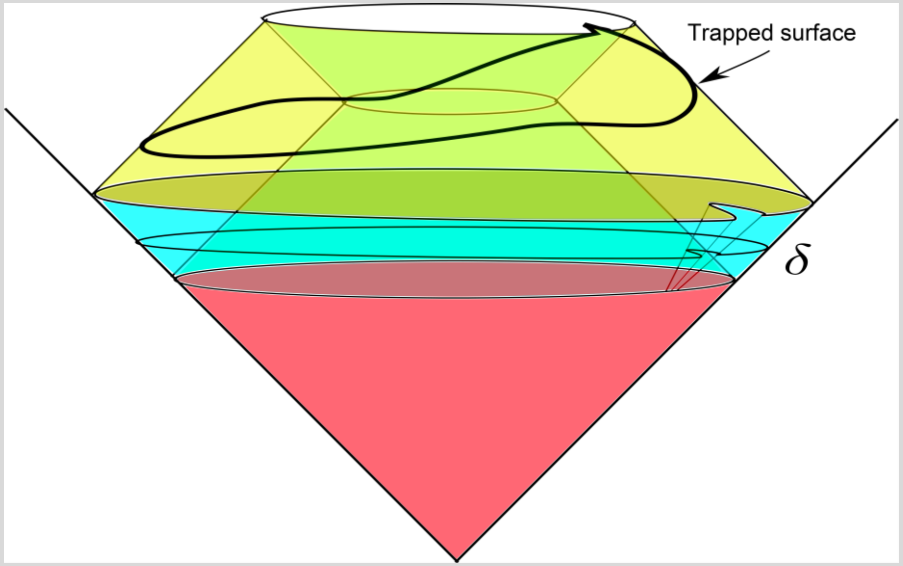
THEOREM

Kl-Luk-Rodnianski (2013)

Result holds true for **non-isotropic** data concentrated near one null geodesic generator.

1. Combines all ingredients in Christodoulou's theorem with a **deformation argument** along incoming null hypersurfaces.
2. Reduces to a simple differential inequality on

$$S_{0,0} = H_0 \cap \underline{H}_0.$$



CONCLUSIONS

RIGIDITY

Completely understood in the static case. In the stationary it is only understood under additional assumptions, to insure closedness to Kerr. General case is wide open.

STABILITY

Remains wide open. We only understand the stability of Minkowski space in full. The mathematical evidence for the general stability of black holes is still scant and is essentially based on linearization. Only the so called *Poor man's linear stability* is now completely understood. There is hope that nonlinear stability in the restrictive class of axial symmetric perturbations could be settled in the near future.

COLLAPSE

Major results have been obtained in recent years, but the entire scope of the problem is far from being exhausted.

CONCLUSIONS

Development of new mathematical methods and strategies to deal with strong gravitational fields.

- **VECTORFIELD METHOD** Powerful mechanism to derive decay estimates for the linearized equations.
- **NULL STRUCTURE** Einstein equations have a unique geometric structure which allows, in conjunction with the vectorfield method, to control the nonlinear equations.
- **LONG TIME CONTROL** Understanding of a general mechanism for long time control of the Einstein equations. Stability of Minkowski space.
- **BLACK HOLE FORMATION** Understanding of a powerful new mechanism (in vacuum, non-isotropic) for the formation of black holes.
- **UNIQUE CONTINUATION** Powerful method to deal with ill posed problem in General Relativity, most importantly in the rigidity problem.