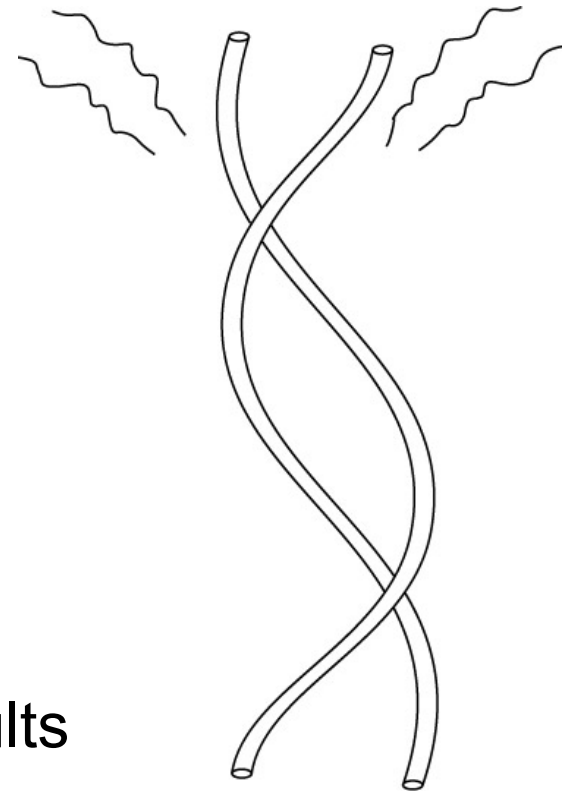


The Problem of Motion in General Relativity

- Solve

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

or $R_{\mu\nu} = 0$



- Taking into account radiation damping effects
- Extract physical (gauge-invariant) results

Early History of the Problem of Motion

Einstein 1912 : geodesic principle $- \int m \sqrt{-g_{\mu\nu}} dx^\mu dx^\nu$

Einstein 1913-1916 post-Minkowskian $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) , h_{\mu\nu} \ll 1$

Einstein, Droste : post-Newtonian $h_{00} \sim h_{ij} \sim \frac{v^2}{c^2} , h_{0i} \sim \frac{v^3}{c^3} , \partial_0 h \sim \frac{v}{c} \partial_i h$


Mercury's perihelion : Einstein-Besso 1913, Droste 1914, Einstein 1915

N-body problem in PN theory: Droste 1916, De Sitter 1916, [Lorentz-Droste 1917], Chazy 1928,1930, Levi-Civita 1937

Early History : Various Approaches

- Post-Newtonian: Droste '16, De Sitter '16, Lorentz-Droste '17, Chazy '28, Levi-Civita '37

- $\nabla_\nu T^{\mu\nu} = 0 ; \quad T^{\mu\nu} = \rho u^\mu u^\nu \Rightarrow \nabla_u u^\mu = 0$



Eddington' 21, ..., Fock '39, Papapetrou '51, Lichnerowicz '55, ... Dixon '64, Bailey-Israel '75, Ehlers-Rudolph '77....

- Fitting a worldtube containing a singularity
Weyl '21, Einstein-Grommer '27, Einstein-Infeld-Hoffman '38,...

- δ -function source: Mathisson'31, ..., Infeld '54

- Regularization / Renormalization issues for point particles

World-tubes : Dirac '38, Bhabha-Harish-Chandra '44, ...

Analytic continuation: Riesz '39, Fremberg '46, Ma '47, ...

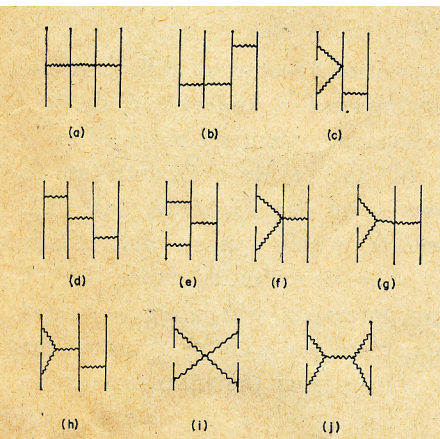
Later History : New Tools

- **Post-Minkowskian**: Bertotti '56, Havas'57, Kerr'59, Bertotti-Plebanski'60, Havas-Goldberg'62, Schmutzer '66, ..., Thorne-Kovacs '75, ... , Rosenblum '78, Westpfahl-Göller '79, Bel et al. '81, Damour '82, Ledvinka-Schäfer-Bicak '08
- **ADM Hamiltonian formalism**: Kimura '61 [1PN], Ohta-Okamura-Kimura-Hiida '74 [2PN], Schäfer '85 [2.5PN], Jaranowski-Schäfer '98 [3PN],.....
- **Reduced Action à la Fokker '29**: Infeld '57, Infeld-Plebanski '60,

- **Quantum S-matrix → potential**

one-graviton exchange : Corinaldesi '56 '71, Barker-Gupta-Haracz 66, Barker-O'Connell 70, Hiida-Okamura 72

Nonlinear: Iwasaki 7 [1PN], Okamura-Ohta-Kimura-Hiida 73[2 PN]



1974-: Discovery of the Hulse-Taylor Binary Pulsar

Many entangled issues and a motivation for improving the methods

- need to reach the radiation-damping level $2.5 \text{ PN} = (v/c)^5$
- need an approach valid for strongly-self gravitating bodies
- need to tackle IR divergent integrals in PN schemes
- need to tackle UV divergent integrals when using δ -sources
- need to include spin effects
- need to derive gauge-invariant, observable effects
- need a relativistic timing formula

1979-1983: The “Quadrupole Controversy”

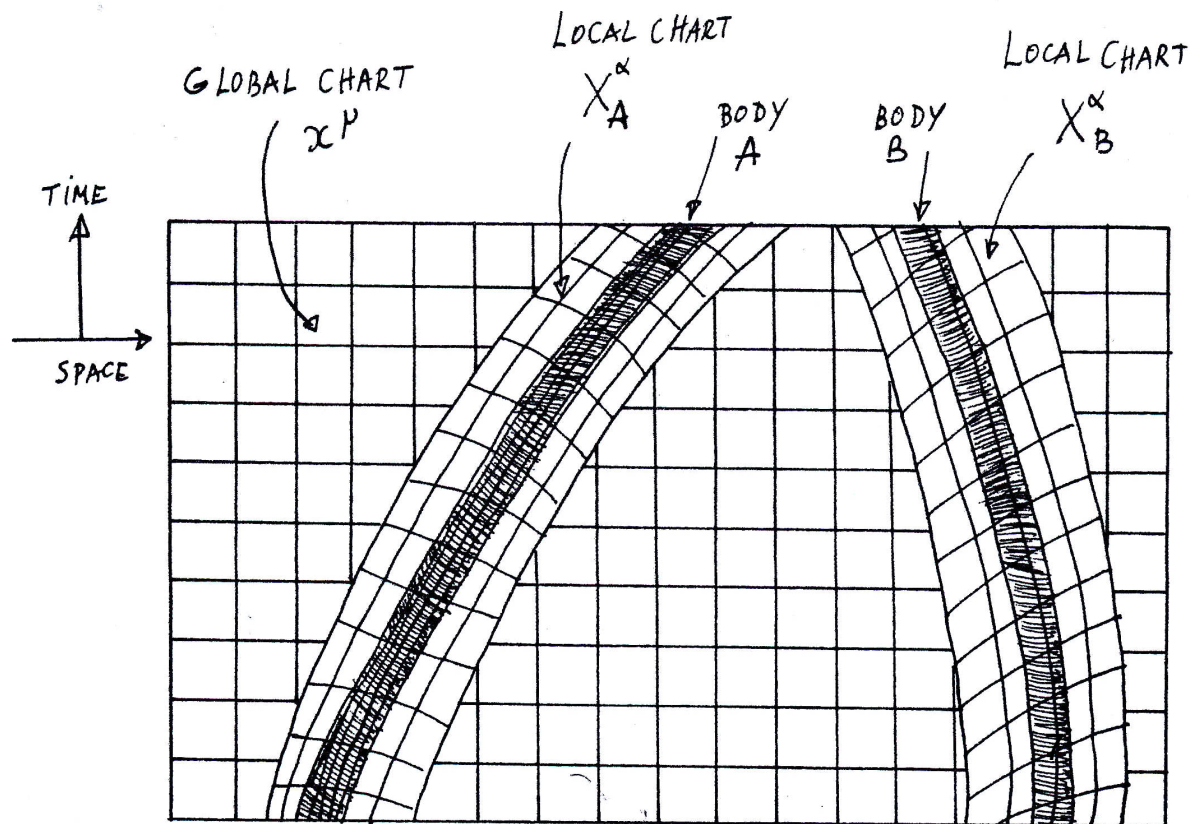
A flurry of (sometimes conflicting) results

- Old results: Eddington, Infeld, Peres, Carmeli, Havas, Hu, Burke, Chandrasekhar,
- New results: Walker-Will 80, Kerlick 80, Schutz 80, Breuer-Rudolph 81, Papapetrou-Linet 81, McCrea 81, Rosenblum 81, Damour-Deruelle 81-82, Futamase 83, Schäfer 85, Kopeikin 85,

New Techniques in the Problem of Motion (1)

- **Multi-chart** approach and **matched asymptotic expansions**: necessary for strongly self-gravitating bodies (NS, BH)
Manasse (Wheeler) '63, Demianski-Grishchuk '74, D'Eath '75, Kates '80, Damour '82

Useful even for weakly self-gravitating bodies, i.e. "relativistic celestial mechanics",
Brumberg-Kopeikin '89,
Damour-Soffel-Xu '91-94



New Techniques in the Problem of Motion (2)

Skeletonization : $T_{\mu\nu} \longrightarrow$ point-masses

delta-functions in GR : Infeld '54, Infeld-Plebanski '60, ..., Eardley '75

QFT's **dimensional regularization** (Bollini-Giambiagi '72, t'Hooft-Veltman '72)

imported in GR (Damour '80, Damour-Jaranowski-Schäfer '01, ...)

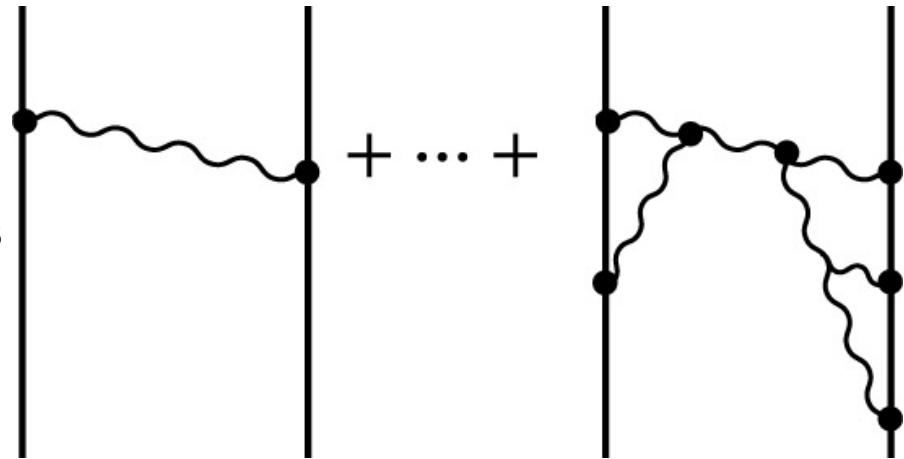
Feynman-like diagrams and
« **Effective Field Theory** » techniques

Bertotti-Plebanski '60,

Damour-Esposito-Farèse '96,

Goldberger-Rothstein '06, Porto '06, Gilmore-Ross' 08, Levi '10,

Foffa-Sturani '11 '13, Levi-Steinhoff '14, '15



Post-Newtonian Equations of Motion [2-body, wo spins]

- 1PN (including v^2/c^2) Lorentz-Droste '17, EIH '38, Eddington-Clark '38
- 2PN (inc. v^4/c^4) Ohta-Okamura-Kimura-Hiida '74, Damour-Deruelle '81
Damour '82, Schäfer '85, Kopeikin '85
- 2.5 PN (inc. v^5/c^5) Damour-Deruelle '81, Damour '82, Schäfer '85,
Kopeikin '85
- 3 PN (inc. v^6/c^6) Jaranowski-Schäfer '98, Blanchet-Faye '00,
Damour-Jaranowski-Schäfer '01, Itoh-Futamase '03,
Blanchet-Damour-Esposito-Farèse '04, Foffa-Sturani '11
- 3.5 PN (inc. v^7/c^7) Iyer-Will '93, Jaranowski-Schäfer '97, Pati-Will '02,
Königsdörffer-Faye-Schäfer '03, Nissanke-Blanchet '05, Itoh '09
- 4PN (inc. v^8/c^8) Jaranowski-Schäfer '13, Foffa-Sturani '13,
Bini-Damour '13, Damour-Jaranowski-Schäfer '14

New feature : non-locality in time

2-body Taylor-expanded N + 1PN + 2PN Hamiltonian

$$H_N(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{1}{2} \frac{Gm_1m_2}{r_{12}} + (1 \leftrightarrow 2)$$

$$c^2 H_{1PN}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8} \frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \left(-12 \frac{\mathbf{p}_1^2}{m_1^2} + 14 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1m_2} + 2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1m_2} \right) \\ + \frac{1}{4} \frac{Gm_1m_2}{r_{12}} \frac{G(m_1 + m_2)}{r_{12}} + (1 \leftrightarrow 2),$$

$$c^4 H_{2PN}(\mathbf{x}_a, \mathbf{p}_a) = \frac{1}{16} \frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \left(5 \frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2} \frac{\mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} - \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} + 5 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right. \\ \left. - 6 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} - \frac{3}{2} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right) \\ + \frac{1}{4} \frac{G^2 m_1 m_2}{r_{12}^2} \left(m_2 \left(10 \frac{\mathbf{p}_1^2}{m_1^2} + 19 \frac{\mathbf{p}_2^2}{m_2^2} \right) - \frac{1}{2} (m_1 + m_2) \frac{27(\mathbf{p}_1 \cdot \mathbf{p}_2) + 6(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \right) \\ - \frac{1}{8} \frac{Gm_1m_2}{r_{12}} \frac{G^2(m_1^2 + 5m_1m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2),$$

2-body Taylor-expanded 3PN Hamiltonian [JS 98, DJS 01]

$$\begin{aligned}
 c^6 H_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = & -\frac{5}{128} \frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32} \frac{Gm_1 m_2}{r_{12}} \left(-14 \frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4 \frac{((\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 4\mathbf{p}_1^2 \mathbf{p}_2^2) \mathbf{p}_1^2}{m_1^4 m_2^2} + 6 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^4 m_2^2} \right. \\
 & - 10 \frac{(\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 + \mathbf{p}_2^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2) \mathbf{p}_1^2}{m_1^4 m_2^2} + 24 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^4 m_2^2} \\
 & + 2 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} + \frac{(7\mathbf{p}_1^2 \mathbf{p}_2^2 - 10(\mathbf{p}_1 \cdot \mathbf{p}_2)^2) (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2^3} \\
 & + \frac{(\mathbf{p}_1^2 \mathbf{p}_2^2 - 2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2) (\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2^3} + 15 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} \\
 & - 18 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3} + 5 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3} \left. + \frac{G^2 m_1 m_2}{r_{12}^2} \left(\frac{1}{16} (m_1 - 27m_2) \frac{(\mathbf{p}_1^2)^2}{m_1^4} \right. \right. \\
 & - \frac{115}{16} m_1 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2} + \frac{1}{48} m_2 \frac{25(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 371\mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} + \frac{17\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{16 m_1^3} + \frac{5 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{12 m_1^3} \\
 & - \frac{1}{8} m_1 \frac{(15\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2) + 11(\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1)) (\mathbf{n}_{12} \cdot \mathbf{p}_1)}{m_1^3 m_2} - \frac{3}{2} m_1 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2} \\
 & + \frac{125}{12} m_2 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2) (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} + \frac{10}{3} m_2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \\
 & - \frac{1}{48} (220m_1 + 193m_2) \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \left. + \frac{G^3 m_1 m_2}{r_{12}^3} \left(-\frac{1}{48} \left(425m_1^2 + \left(473 - \frac{3}{4} \pi^2 \right) m_1 m_2 + 150m_2^2 \right) \frac{\mathbf{p}_1^2}{m_1^2} \right. \right. \\
 & + \frac{1}{16} \left(77(m_1^2 + m_2^2) + \left(143 - \frac{1}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \frac{1}{16} \left(20m_1^2 - \left(43 + \frac{3}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} \\
 & + \frac{1}{16} \left(21(m_1^2 + m_2^2) + \left(119 + \frac{3}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \\
 & \left. + \frac{1}{8} \frac{G^4 m_1 m_2^3}{r_{12}^4} \left(\left(\frac{227}{3} - \frac{21}{4} \pi^2 \right) m_1 + m_2 \right) + (1 \leftrightarrow 2).
 \end{aligned}$$

2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014]

$$\begin{aligned}
 c^8 H_{4\text{PN}}^{\text{local}}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{7(\mathbf{p}_1^2)^5}{256m_1^9} + \frac{Gm_1m_2}{r_{12}} H_{48}(\mathbf{x}_a, \mathbf{p}_a) + \frac{G^2m_1m_2}{r_{12}^2} m_1 H_{46}(\mathbf{x}_a, \mathbf{p}_a) \\
 &+ \frac{G^3m_1m_2}{r_{12}^3} (m_1^2 H_{441}(\mathbf{x}_a, \mathbf{p}_a) + m_1m_2 H_{442}(\mathbf{x}_a, \mathbf{p}_a)) \\
 &+ \frac{G^4m_1m_2}{r_{12}^4} (m_1^3 H_{421}(\mathbf{x}_a, \mathbf{p}_a) + m_1^2m_2 H_{422}(\mathbf{x}_a, \mathbf{p}_a)) \\
 &+ \frac{G^5m_1m_2}{r_{12}^5} H_{40}(\mathbf{x}_a, \mathbf{p}_a) + (1 \leftrightarrow 2), \tag{A3}
 \end{aligned}$$

$$\begin{aligned}
 H_{48}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{45(\mathbf{p}_1^2)^4}{128m_1^8} - \frac{9(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64m_1^6m_2^2} + \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64m_1^6m_2^2} - \frac{9(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{16m_1^6m_2^2} \\
 &- \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{32m_1^6m_2^2} + \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{21(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{35(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{256m_1^6m_2^2} \\
 &+ \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2}{128m_1^6m_2^2} + \frac{33(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256m_1^6m_2^2} - \frac{85(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^6m_2^2} \\
 &- \frac{45(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{128m_1^6m_2^2} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^6m_2^2} + \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^6m_2^2} \\
 &+ \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^6m_2^2} - \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{64m_1^6m_2^2} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{64m_1^6m_2^2} + \frac{55(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_2^2}{256m_1^6m_2^2} \\
 &- \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128m_1^6m_2^2} - \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256m_1^6m_2^2} - \frac{23(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{256m_1^6m_2^2} \\
 &+ \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{128m_1^6m_2^2} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{256m_1^6m_2^2} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{64m_1^6m_2^2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64m_1^6m_2^2} \\
 &- \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{4m_1^6m_2^2} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16m_1^6m_2^2} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{64m_1^6m_2^2} + \frac{21(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64m_1^6m_2^2} \\
 &- \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32m_1^6m_2^2} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{4m_1^6m_2^2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{16m_1^6m_2^2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{16m_1^6m_2^2} \\
 &- \frac{\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{32m_1^6m_2^2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64m_1^6m_2^2} - \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32m_1^6m_2^2} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128m_1^6m_2^2}, \tag{A4a}
 \end{aligned}$$

$$\begin{aligned}
 H_{46}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{369(\mathbf{n}_{12} \cdot \mathbf{p}_1)^6}{160m_1^6} - \frac{889(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4\mathbf{p}_1^2}{192m_1^6} + \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64m_1^6} - \frac{549(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{128m_1^6m_2} \\
 &+ \frac{67(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{16m_1^6m_2} - \frac{167(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2}{128m_1^6m_2} + \frac{1547(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^6m_2} - \frac{851(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{128m_1^6m_2} \\
 &+ \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^6m_2} + \frac{3263(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{1280m_1^6m_2^2} + \frac{1067(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{480m_1^6m_2^2} - \frac{4567(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840m_1^6m_2^2} \\
 &- \frac{3571(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{320m_1^6m_2^2} + \frac{3073(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{480m_1^6m_2^2} + \frac{4349(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{1280m_1^6m_2^2} \\
 &- \frac{3461\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{3840m_1^6m_2^2} + \frac{1673(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4\mathbf{p}_2^2}{1920m_1^6m_2^2} - \frac{1999(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840m_1^6m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840m_1^6m_2^2} - \frac{13(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{8m_1^6m_2^2} \\
 &+ \frac{191(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2}{192m_1^6m_2^2} - \frac{19(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^6m_2^2} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^6m_2^2} \\
 &+ \frac{11(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{192m_1^6m_2^2} + \frac{77(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{96m_1^6m_2^2} + \frac{233(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_2^2}{96m_1^6m_2^2} - \frac{47(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^6m_2^2} \\
 &+ \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{384m_1^6m_2^2} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{384m_1^6m_2^2} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4}{4m_1^6m_2^2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4\mathbf{p}_1^2}{4m_1^6m_2^2} \\
 &- \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3(\mathbf{p}_1 \cdot \mathbf{p}_2)}{2m_1^6m_2^2} - \frac{21(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16m_1^6m_2^2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{6m_1^6m_2^2} + \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48m_1^6m_2^2} \\
 &- \frac{133(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{24m_1^6m_2^2} - \frac{77(\mathbf{p}_1 \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{96m_1^6m_2^2} + \frac{197(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96m_1^6m_2^2} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48m_1^6m_2^2} + \frac{13(\mathbf{p}_2^2)^3}{8m_2^6}, \tag{A4b}
 \end{aligned}$$

$$\begin{aligned}
 H_{441}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{5027(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{384m_1^4} - \frac{22993(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2}{960m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152m_1^4} - \frac{3191(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{640m_1^3m_2} \\
 &+ \frac{28561(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{1920m_1^3m_2} + \frac{8777(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^3m_2} + \frac{752969\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{28800m_1^3m_2} \\
 &- \frac{16481(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{960m_1^3m_2^2} + \frac{94433(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{4800m_1^3m_2^2} - \frac{103957(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{2400m_1^3m_2^2} \\
 &+ \frac{791(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{400m_1^3m_2^2} + \frac{26627(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_2^2}{1600m_1^3m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800m_1^3m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32m_2^3}, \tag{A4c}
 \end{aligned}$$

$$\begin{aligned}
 H_{442}(\mathbf{x}_a, \mathbf{p}_a) &= \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200} \right) \frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{m_1^4} \\
 &+ \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600} \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^3m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600} \right) \frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^3m_2^2} \\
 &+ \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^3m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3m_2^2} \\
 &- \left(\frac{30383}{960} + \frac{36405\pi^2}{16384} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384} \right) \frac{\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3m_2^2} \\
 &+ \left(\frac{2369}{60} + \frac{35655\pi^2}{16384} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3m_2} + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{m_1^3m_2} \\
 &+ \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3m_2}, \tag{A4d}
 \end{aligned}$$

$$H_{421}(\mathbf{x}_a, \mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800m_1^2} - \frac{91(\mathbf{p}_1 \cdot \mathbf{p}_2)}{8m_1m_2} + \frac{105\mathbf{p}_2^2}{32m_2^2} - \frac{9841(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{1600m_1^2} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{2m_1m_2}, \tag{A4e}$$

$$\begin{aligned}
 H_{422}(\mathbf{x}_a, \mathbf{p}_a) &= \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152} \right) \frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600} \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192} \right) \frac{\mathbf{p}_2^2}{m_2^2} \\
 &+ \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} + \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1m_2} \\
 &+ \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_2^2}, \tag{A4f}
 \end{aligned}$$

$$H_{40}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400} \right) m_1^3m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200} \right) m_1^2m_2^2. \tag{A4g}$$

$$\begin{aligned}
 H_{4\text{PN}}^{\text{nonloc}}(t) &= -\frac{1}{5} \frac{G^2M}{c^8} I_{ij}^{(3)}(t) \\
 &\times \text{Pf}_{2r_{12}/c} \int_{-\infty}^{+\infty} \frac{dv}{|v|} I_{ij}^{(3)}(t+v),
 \end{aligned}$$

Gravitational Radiation

1916, 1918 Einstein : h_+ , h_x and **quadrupole formula**

Relativistic, **multipolar extensions** of LO quadrupole radiation :

Mathews '62,

Bonnor-Rotenberg '66,

Campbell-Morgan '71,

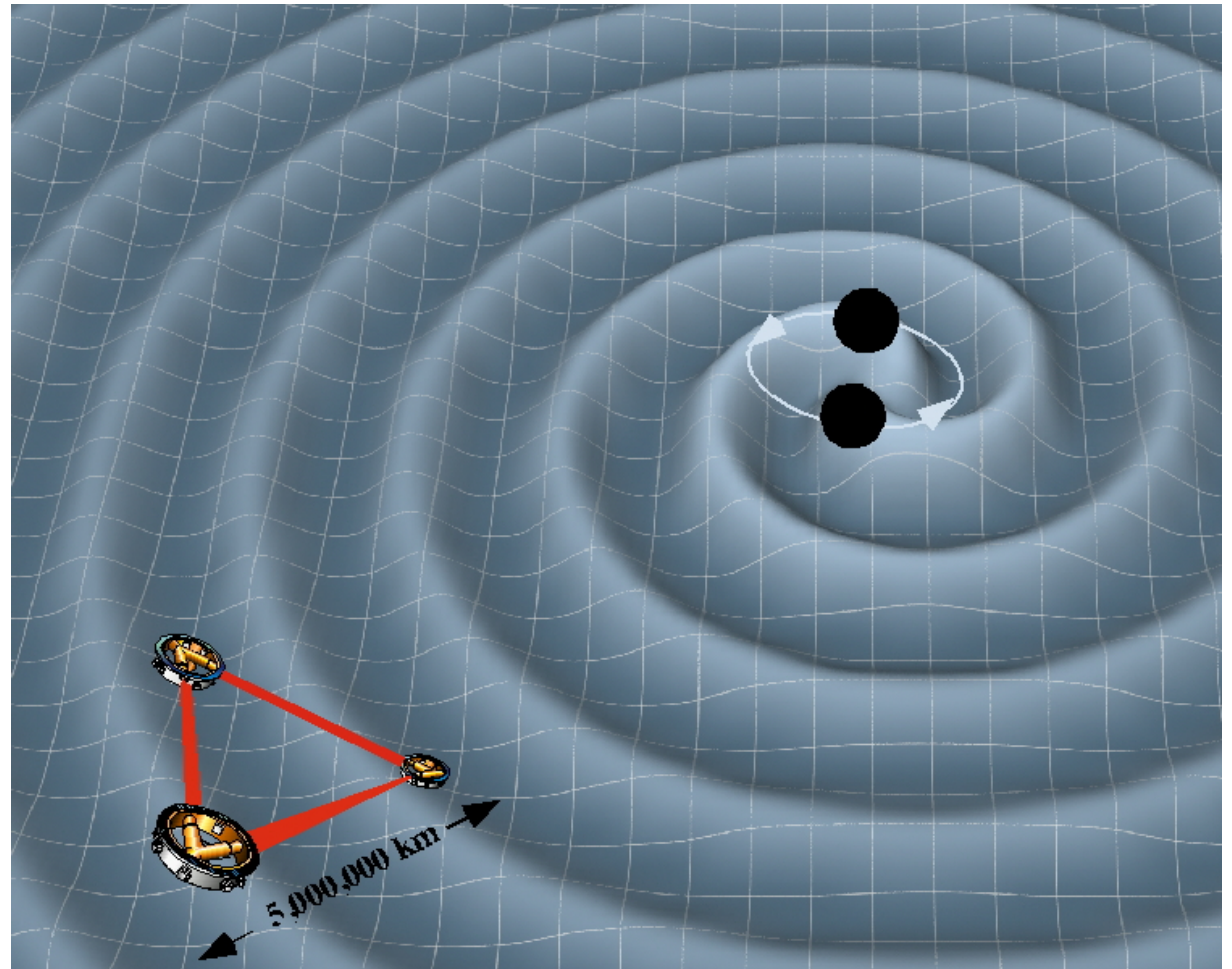
Campbell et al '75,

Thorne '80,

Blanchet-Damour '86,

Damour-Iyer '91,

Will et al '00



Taylor-expanded 3.5PN waveform

Blanchet, Iyer, Joguet 02, Blanchet, Damour, Esposito-Farese, Iyer 04, Kidder 07, Blanchet et al. 08, Faye et al '12, Bohé, Marsat, Blanchet, Buonanno 13-15

$$\begin{aligned} h^{22} = & -8\sqrt{\frac{\pi}{5}} \frac{G\nu m}{c^2 R} e^{-2i\phi} x \left\{ 1 - x \left(\frac{107}{42} - \frac{55}{42} \nu \right) + x^{3/2} \left[2\pi + 6i \ln\left(\frac{x}{x_0}\right) \right] - x^2 \left(\frac{2173}{1512} + \frac{1069}{216} \nu - \frac{2047}{1512} \nu^2 \right) \right. \\ & - x^{5/2} \left[\left(\frac{107}{21} - \frac{34}{21} \nu \right) \pi + 24i\nu + \left(\frac{107i}{7} - \frac{34i}{7} \nu \right) \ln\left(\frac{x}{x_0}\right) \right] \\ & + x^3 \left[\frac{27\,027\,409}{646\,800} - \frac{856}{105} \gamma_E + \frac{2}{3} \pi^2 - \frac{1712}{105} \ln 2 - \frac{428}{105} \ln x \right. \\ & \left. \left. - 18 \left[\ln\left(\frac{x}{x_0}\right) \right]^2 - \left(\frac{278\,185}{33\,264} - \frac{41}{96} \pi^2 \right) \nu - \frac{20\,261}{2772} \nu^2 + \frac{114\,635}{99\,792} \nu^3 + \frac{428i}{105} \pi + 12i\pi \ln\left(\frac{x}{x_0}\right) \right] + O(\epsilon^{7/2}) \right\}, \end{aligned}$$

$$x = (M\Omega)^{2/3} \sim v^2/c^2$$

$$M = m_1 + m_2$$

$$\nu = m_1 m_2 / (m_1 + m_2)^2$$

Gravitational Radiation Damping : Two Views

Balance between the E and J of the system and GW losses at infinity

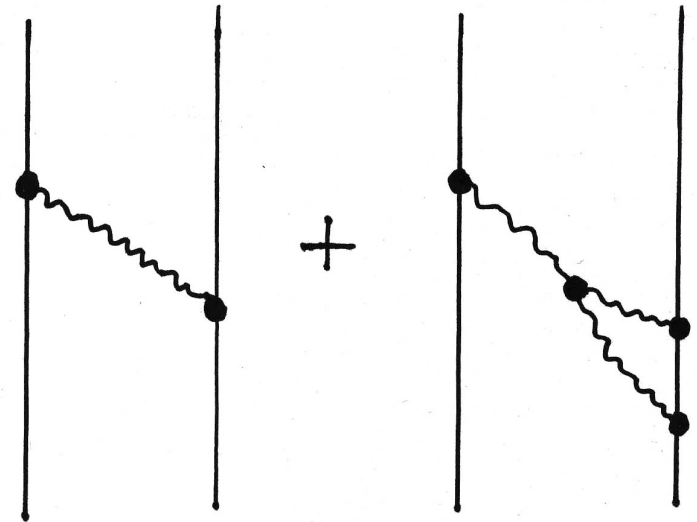
(Landau-Lifshitz, Fock, Peters-Mathews)

Radiation Reaction : « Laplace-Eddington effect »

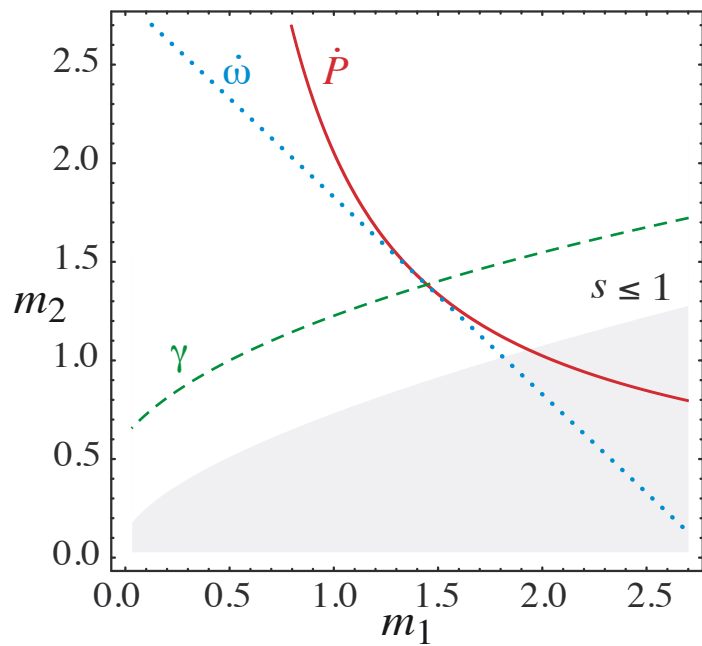
Eddington 1924 : « The problem of the double star is more difficult ; we should have to take account of the effect of the gravitational field in disturbing the propagation of its own potentials and we cannot be sure that even the sign of (1) [quadrupole energy loss] is correct. »

1974 Hulse-Taylor binary pulsar : new motivation

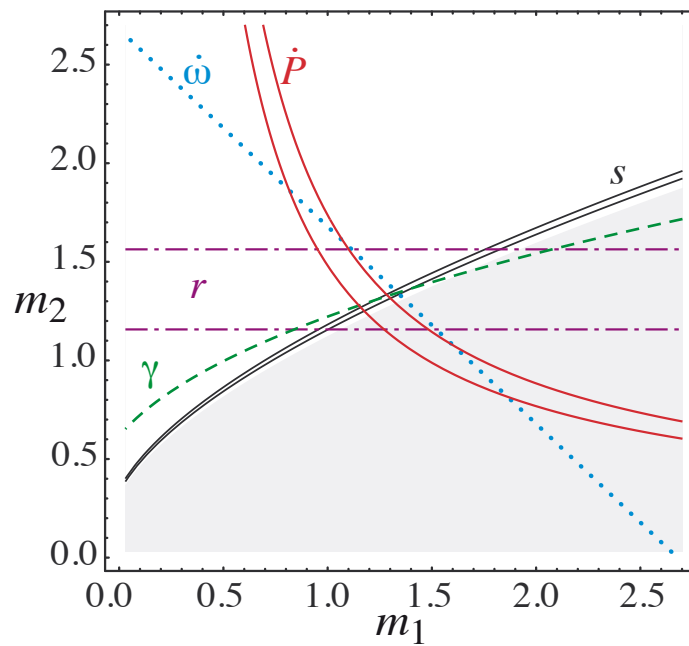
2.5 PN equation of motion (T.D. '82) showed that binary pulsar data give a **direct** proof that the gravitational interaction propagates at velocity c à la GR



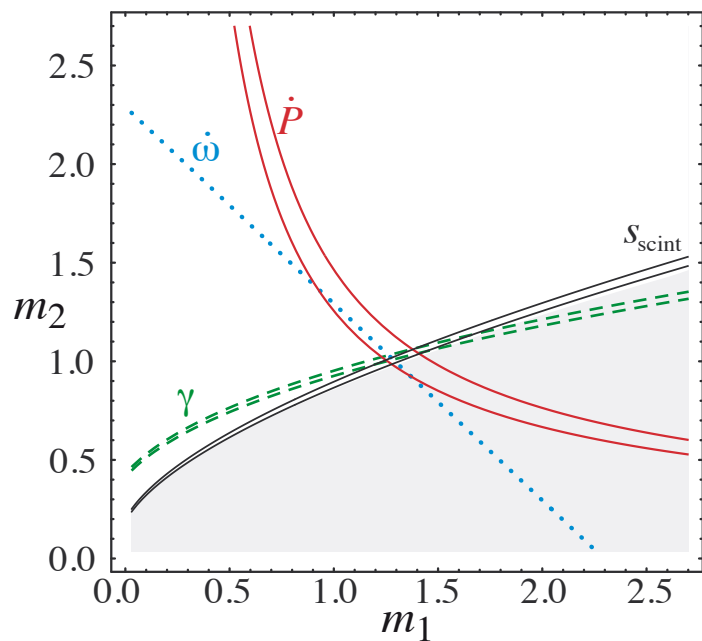
PSR B1913+16



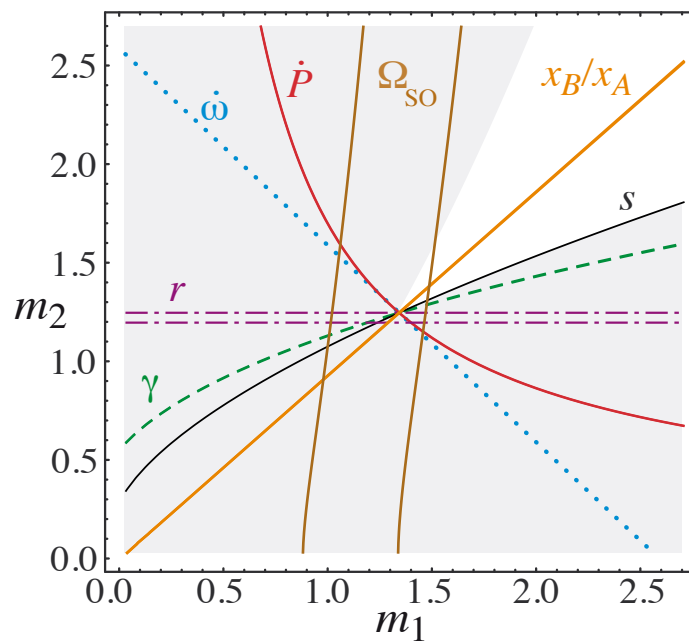
PSR B1534+12



PSR J1141-6545

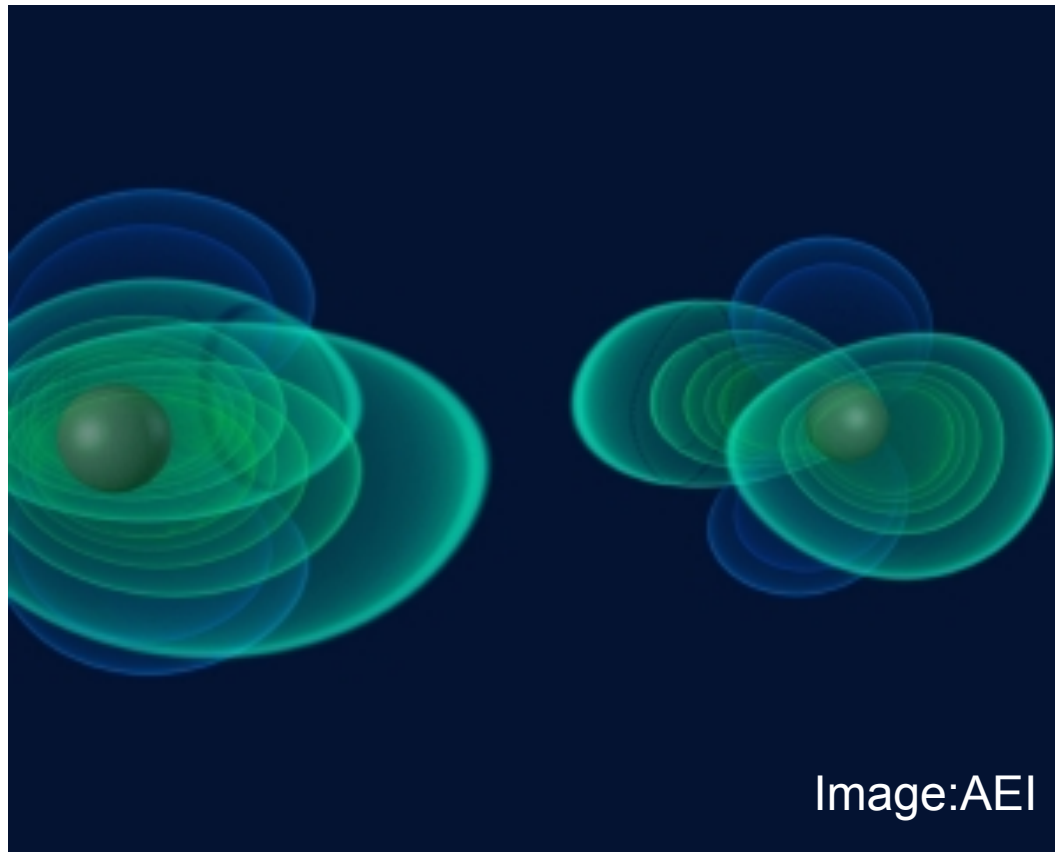


PSR J0737-3039



Numerical Relativity

2005 breakthrough (Pretorius '05, Campanelli et al '06, Baker et al '06, Diener et al '06, Brügmann et al, Boyle et al, Rezzolla, Hinder, Pollney, Reisswig, Mroué, Scheel, Szilagyi, Pfeiffer, ...) many impressive results for binary systems of BH or NS (Shibata, Hotokezaka, Rezzolla, Baiotti, Brügmann, Bernuzzi, Hilditch,)



Caltech-Cornell-CITA catalog of gravitational waveforms

A catalog of 171 high-quality binary black-hole simulations for gravitational-wave astronomy [\[arXiv: 1304.6077\]](https://arxiv.org/abs/1304.6077)

Abdul H. Mroué,¹ Mark A. Scheel,² Béla Szilágyi,² Harald P. Pfeiffer,¹ Michael Boyle,³ Daniel A. Hemberger,³ Lawrence E. Kidder,³ Geoffrey Lovelace,^{4,2} Sergei Ossokine,^{1,5} Nicholas W. Taylor,² Anil Zenginoğlu,² Luisa T. Buchman,² Tony Chu,¹ Evan Foley,⁴ Matthew Giesler,⁴ Robert Owen,⁶ and Saul A. Teukolsky³

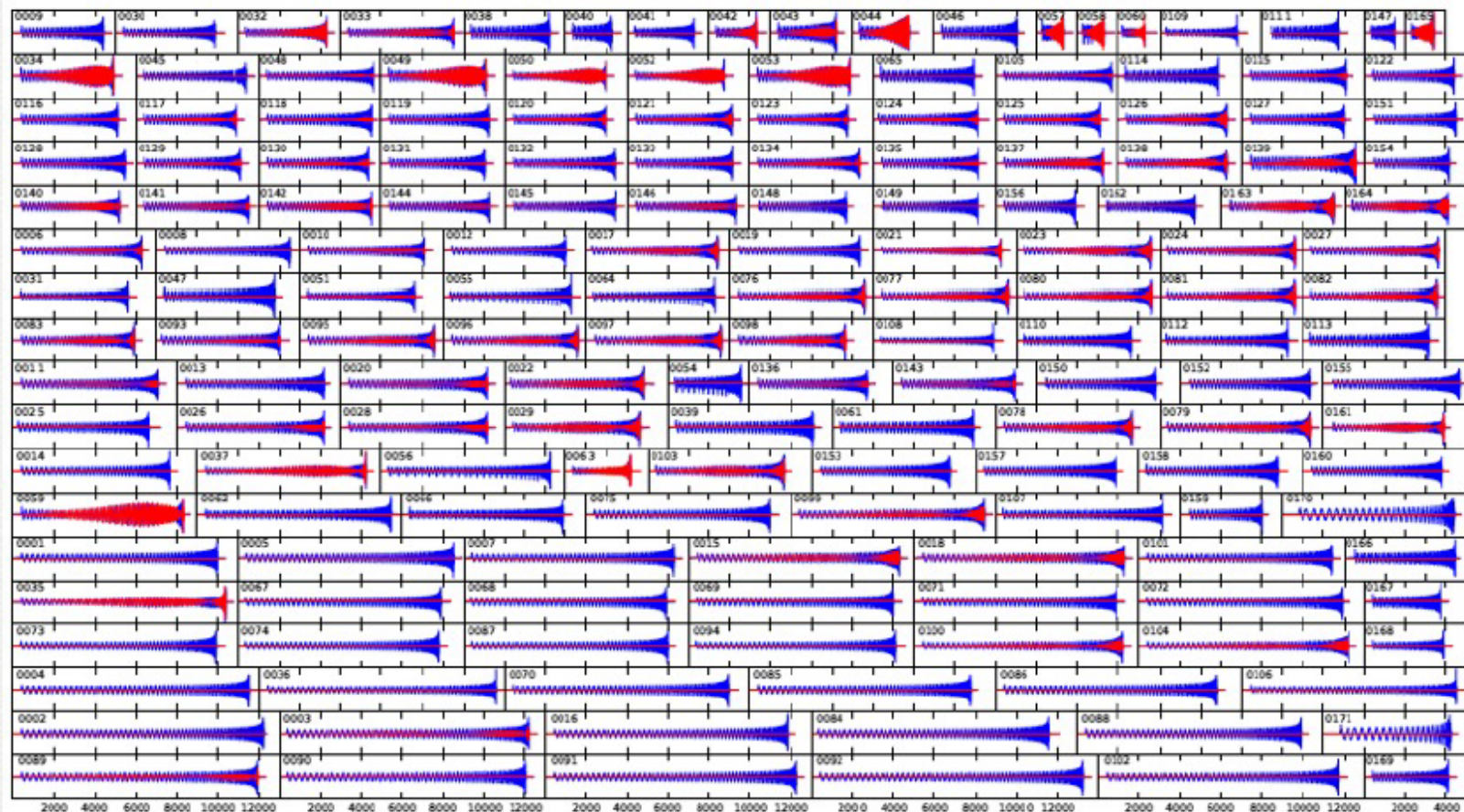


FIG. 3: Waveforms from all simulations in the catalog. Shown here are h_+ (blue) and h_x (red) in a sky direction parallel to the initial orbital plane of each simulation. All plots have the same horizontal scale, with each tick representing a time interval of $2000M$, where M is the total mass.

Gravitational Self-Force Theory : $m_1 \ll m_2$

- First-order beyond the geodesic test-mass dynamics :

one-loop in BH background

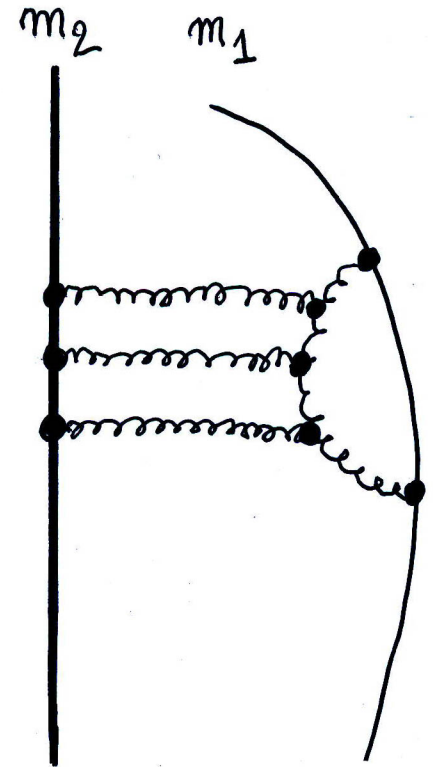
DeWitt-Brehme '60, Mino-Sasaki-Tanaka '97,

Quinn-Wald '97, ..., Barack-Ori '02,

Mino-Nakano-Sasaki '03, Detweiler-Whiting '03

- Uses BH perturbation theory : Regge-Wheeler '57,

Zerilli '70, Teukolsky '72, Mano-Suzuki-Takasugi '96



- Motivated by LISA-type astrophysics, but useful for getting the terms linear in $q = m_1/m_2$ either to (very) high PN orders, or numerically.

Gravitational Self-Force Theory : $m_1 \ll m_2$

- Analytical high-PN results : Blanchet-Detweiler-LeTiec-Whiting '10, Damour '10, Blanchet et al '10, LeTiec et al '12, Bini-Damour '13-15, Kavanagh-Ottewill-Wardell '15
- (gauge-invariant) Numerical results : Detweiler '08, Barack-Sago '09, Blanchet-Detweiler-LeTiec-Whiting '10, Barack-Damour-Sago '10, Shah-Friedman-Keidl '12, Dolan et al '14, Nolan et al '15, ...
- Analytical PN results from high-precision (hundreds to thousands of digits !) numerical results : Shah-Friedman-Whiting '14, Johnson-McDaniel-Shah-Whiting '15

GSF: Analytical High PN results (1) (Bini-Damour '15)

$$\begin{aligned}
 a_{10}^c = & \frac{18605478842060273}{7079830758000} \ln(2) - \frac{1619008}{405} \zeta(3) - \frac{21339873214728097}{1011404394000} \gamma \\
 & + \frac{27101981341}{100663296} \pi^6 - \frac{6236861670873}{125565440} \ln(3) + \frac{360126}{49} \ln(2) \ln(3) + \frac{180063}{49} \ln(3)^2 \\
 & - \frac{121494974752}{9823275} \ln(2)^2 - \frac{24229836023352153}{549755813888} \pi^4 + \frac{1115369140625}{124540416} \ln(5) + \frac{96889010407}{277992000} \ln(7) \\
 & + \frac{75437014370623318623299}{18690753201120000} - \frac{60648244288}{9823275} \ln(2) \gamma + \frac{200706848}{280665} \gamma^2 \\
 & + \frac{11980569677139}{2306867200} \pi^2 + \frac{360126}{49} \gamma \ln(3),
 \end{aligned}$$

$$a_{10}^{\ln} = -\frac{21275143333512097}{2022808788000} + \frac{200706848}{280665} \gamma - \frac{30324122144}{9823275} \ln(2) + \frac{180063}{49} \ln(3),$$

$$a_{10}^{\ln^2} = \frac{50176712}{280665},$$

$$a_{10.5}^c = -\frac{185665618769828101}{24473489040000} \pi + \frac{377443508}{77175} \ln(2) \pi + \frac{2414166668}{1157625} \pi \gamma - \frac{5846788}{11025} \pi^3 - \frac{246402}{343} \pi \ln(3),$$

$$a_{10.5}^{\ln} = \frac{1207083334}{1157625} \pi.$$

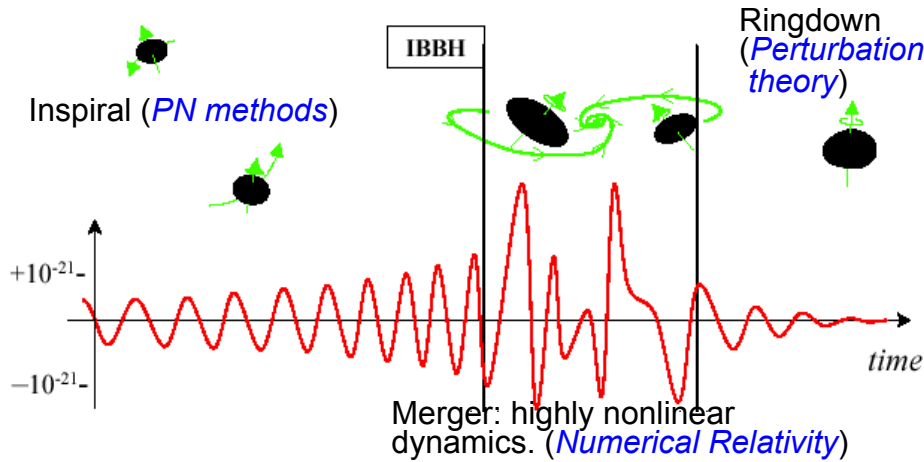
GSF: Analytical High PN results (2)

(Kavanagh et al '15)

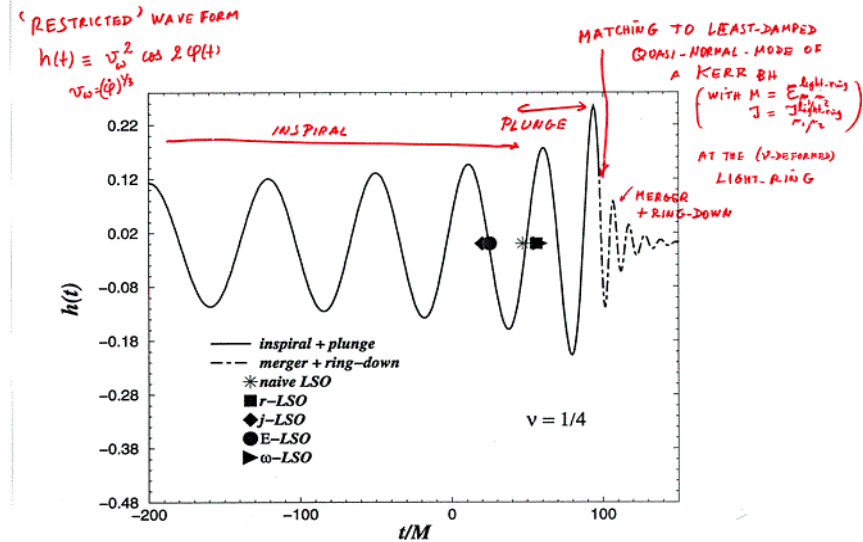
$$\begin{aligned}
 C_{15} = & -\frac{2069543450583769619340376724}{325477442086506084375} \zeta(3) + \frac{65195026298245007936}{22370298575625} \gamma \zeta(3) - \frac{5049442304}{25725} \gamma^2 \zeta(3) + \frac{1262360576}{15435} \pi^2 \zeta(3) \\
 & + \frac{171722752}{441} \zeta(3)^2 + \frac{1613866959570176}{496621125} \zeta(5) - \frac{343445504}{441} \gamma \zeta(5) - \frac{146997248}{105} \zeta(7) + \frac{56314978304}{385875} \zeta(3) \log^2(2) \\
 & - \frac{106445664}{343} \zeta(3) \log^2(3) + \frac{151670998244849797696}{22370298575625} \zeta(3) \log(2) - \frac{190336581632}{1157625} \gamma \zeta(3) \log(2) \\
 & + \frac{28863591064624341}{4909804900} \zeta(3) \log(3) - \frac{212891328}{343} \gamma \zeta(3) \log(3) - \frac{212891328}{343} \zeta(3) \log(2) \log(3) - \frac{77186767578125}{19876428} \zeta(3) \log(5) \\
 & - \frac{2039263232}{3675} \zeta(5) \log(2) - \frac{49128768}{49} \zeta(5) \log(3) + \frac{298267427515018397019736592175289419501391539444290849}{6587612222544653226142468405031917319531250} \\
 & - \frac{6807661768453637768313286948060329087501419}{704310948124803722562607729544062500} \gamma + \frac{1598346944412603247831006289829388}{526171715038677033591890625} \gamma^2 - \frac{100764714621597102764416}{335890033113009375} \gamma^3 \\
 & + \frac{461219496448}{72930375} \gamma^4 - \frac{28338275082077591587855063450276303790065762907243197}{999703155845143418115744045792755712000000} \pi^2 + \frac{25191178655399275691104}{67178006622601875} \gamma \pi^2 \\
 & - \frac{230609748224}{14586075} \gamma^2 \pi^2 + \frac{105480323357757226894713787760391180776248036241}{304245354831316028025099055320268800000} \pi^4 + \frac{1262360576}{385875} \gamma \pi^4 \\
 & - \frac{6208472839612966972691457131143}{266930151354100246118400} \pi^6 + \frac{3573178781920929118281329}{151996487423754240} \pi^8 - \frac{10136323685888}{72930375} \log^4(2) + \frac{38438712}{2401} \log^4(3) \\
 & - \frac{177896086126482679647872}{54963823600310625} \log^3(2) - \frac{89686013106176}{364651875} \gamma \log^3(2) + \frac{153754848}{2401} \log^3(2) \log(3) \\
 & - \frac{131463845322790269123}{245735735245000} \log^3(3) + \frac{153754848}{2401} \gamma \log^3(3) + \frac{153754848}{2401} \log(2) \log^3(3) + \frac{11933074267578125}{51161925672} \log^3(5) \\
 & + \frac{3878258674166628974595420635200204}{189421817413923732093080625} \log^2(2) - \frac{3440856379914601692151168}{1007670099393028125} \gamma \log^2(2) - \frac{16582891400192}{121550625} \gamma^2 \log^2(2) \\
 & + \frac{4145722850048}{72930375} \pi^2 \log^2(2) - \frac{523697163373483905609}{245735735245000} \log^2(2) \log(3) + \frac{461264544}{2401} \gamma \log^2(2) \log(3) \\
 & + \frac{45454535766189065888302299261759}{6569728226789883034880000} \log^2(3) - \frac{394391535968370807369}{245735735245000} \gamma \log^2(3) + \frac{230632272}{2401} \gamma^2 \log^2(3) \\
 & - \frac{96096780}{2401} \pi^2 \log^2(3) - \frac{437493411770075173449}{245735735245000} \log(2) \log^2(3) + \frac{461264544}{2401} \gamma \log(2) \log^2(3) \\
 & + \frac{230632272}{2401} \log^2(2) \log^2(3) + \frac{11933074267578125}{17053975224} \log^2(2) \log(5) - \frac{2505842696993145943705498046875}{402136320895332222431232} \log^2(5) \\
 & + \frac{11933074267578125}{17053975224} \gamma \log^2(5) + \frac{11933074267578125}{17053975224} \log(2) \log^2(5) + \frac{47929508316470415142010251}{56464635170211840000} \log^2(7) \\
 & - \frac{181636067216895220421537747685253699734494659}{6338798533123233503063469565896562500} \log(2) + \frac{74203662155219108543799531653010136}{4735545435348093302327015625} \gamma \log(2) \\
 & - \frac{1482169326522492515499392}{1007670099339028125} \gamma^2 \log(2) - \frac{4905667647488}{364651875} \gamma^3 \log(2) + \frac{371228115490667668451168}{604602059603416875} \pi^2 \log(2) \\
 & + \frac{1226416911872}{72930375} \gamma \pi^2 \log(2) + \frac{23792072704}{17364375} \pi^4 \log(2) - \frac{4141158375397180302387095124935855747727}{108266631596274488880198656000000} \log(3) \\
 & + \frac{9459358001131575454332055276239}{691550339662092951040000} \gamma \log(3) - \frac{394391535968370807369}{245735735245000} \gamma^2 \log(3) + \frac{153754848}{2401} \gamma^3 \log(3) \\
 & + \frac{131463845322790269123}{196588588196000} \pi^2 \log(3) - \frac{192193560}{2401} \gamma \pi^2 \log(3) + \frac{8870472}{1715} \pi^4 \log(3) \\
 & + \frac{214411501060211389845962927148381}{13139456453579766069760000} \log(2) \log(3) - \frac{437493411770075173449}{122867867622500} \gamma \log(2) \log(3) \\
 & + \frac{461264544}{2401} \gamma^2 \log(2) \log(3) - \frac{192193560}{2401} \pi^2 \log(2) \log(3) + \frac{978612948501709853277095576118865234375}{17942749191956127021132384903168} \log(5) \\
 & - \frac{2505842696993145943705498046875}{201068160447666111215616} \gamma \log(5) + \frac{11933074267578125}{17053975224} \gamma^2 \log(5) - \frac{59665371337890625}{204647702688} \pi^2 \log(5) \\
 & - \frac{2505842696993145943705498046875}{201068160447666111215616} \log(2) \log(5) + \frac{11933074267578125}{8526987612} \gamma \log(2) \log(5) \\
 & - \frac{5858006173792308915665113013914648081}{323919193207512802977792000000} \log(7) + \frac{47929508316470415142010251}{28232317585105920000} \gamma \log(7) \\
 & + \frac{47929508316470415142010251}{28232317585105920000} \log(2) \log(7) + \frac{7400249944258160101211}{65676344832000000} \log(11),
 \end{aligned}$$

Templates for GWs from BBH coalescence

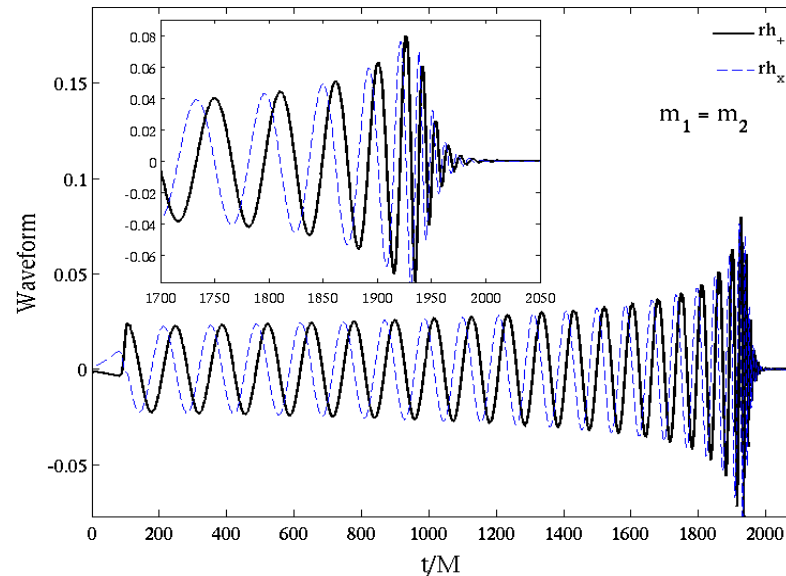
(Brady, Craighton, Thorne 1998)



(Buonanno & Damour 2000)

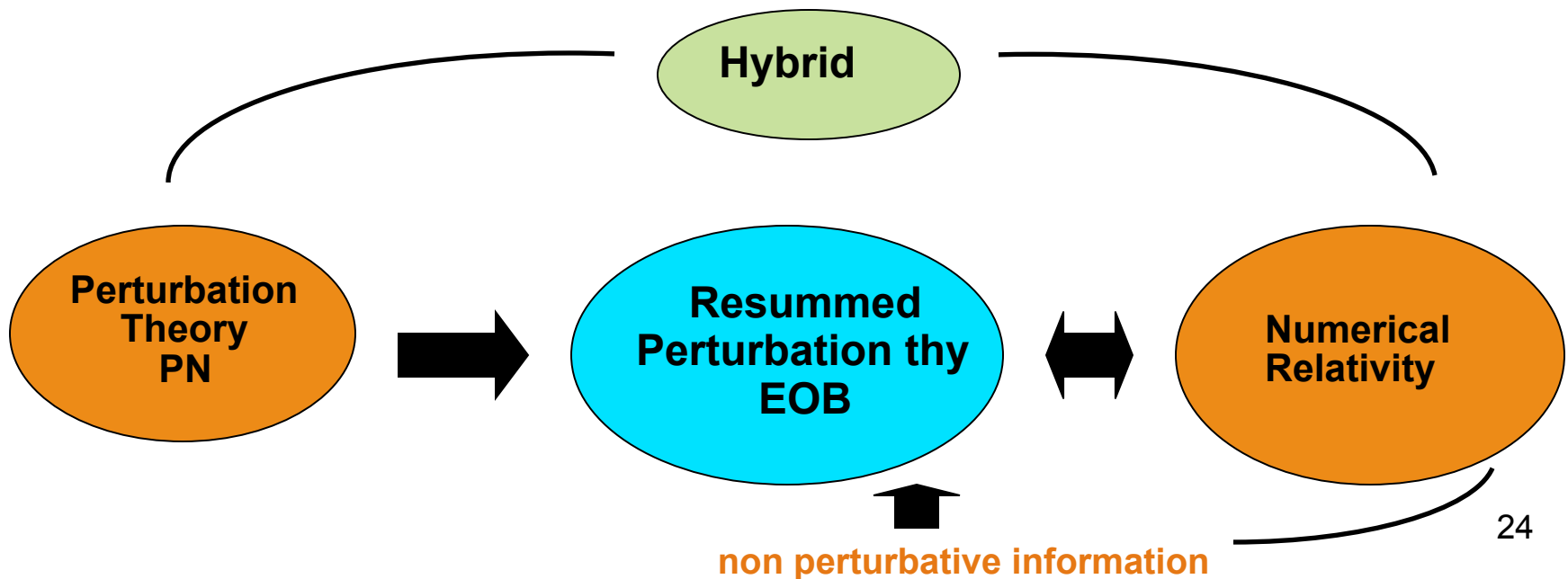


Numerical Relativity, the 2005 breakthrough:
Pretorius, Campanelli et al., Baker et al. ...



Importance of an analytical formalism

- **Theoretical:** physical understanding of the coalescence process, especially in complicated situations (arbitrary spins)
- **Practical:** need many thousands of accurate GW templates for detection & data analysis; need some “analytical” representation of waveform templates as $f(m_1, m_2, \mathbf{S}_1, \mathbf{S}_2)$
- Solution: **synergy between analytical & numerical relativity**



An improved analytical approach

EFFECTIVE ONE BODY (EOB) approach to the two-body problem

Buonanno,Damour 99

Buonanno,Damour 00

Damour, Jaranowski,Schäfer 00

Damour 01, Buonanno, Chen, Damour 05,

Barausse, Buonanno, 10 ... Damour,Nagar 14

Damour, Nagar 07, Damour, Iyer, Nagar 08

Buonanno, Cook, Pretorius 07, Buonanno, Pan, Taracchini,...

Damour, Nagar 10

Bini, Damour 13, Damour, Jaranowski, Schäfer 15

(2 PN Hamiltonian)

(Rad.Reac. full waveform)

(3 PN Hamiltonian)

(spin)

(factorized waveform)

(comparison to NR)

(tidal effects)

(4 PN Hamiltonian)

EOB approach to the two-body problem

1. **Replace** the conservative PN dynamics of two bodies, m_1, m_2 by the

dynamics of a particle of mass $\mu = m_1 m_2 / (m_1 + m_2)$

moving in some effective metric $g_{\mu\nu}^{eff}$ which is a **deformation of a**

Schwarzschild (or Kerr) metric of mass $M = m_1 + m_2$ by the parameter :

$$\nu = \mu / M = m_1 m_2 / (m_1 + m_2)^2$$

2. **Resum** the PN waveform h^{resum}

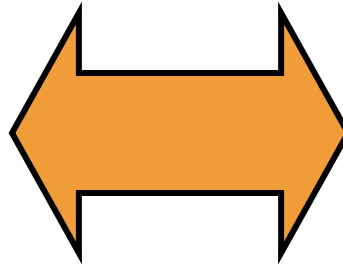
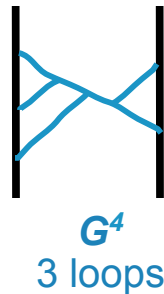
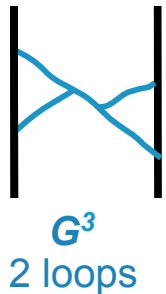
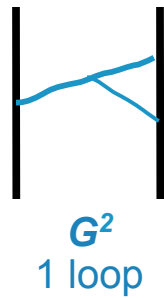
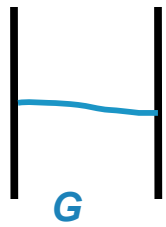
3. **Add** a radiation-reaction force $\mathcal{F}[h^{resum}]$

4. **Complete** the inspiral + plunge + merger gravitational waveform emitted by the EOB dynamics by attaching a **QNM-constructed** ringdown waveform

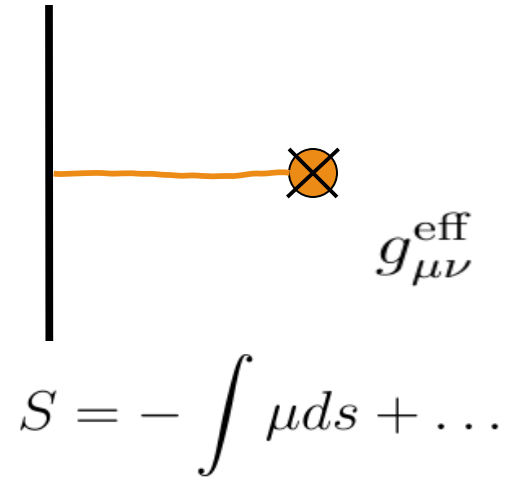
(Vishveshwara 70, Davis-Ruffini-Press-Price 71, Davis-Ruffini-Tiomno 72)

Real dynamics versus Effective dynamics

Real dynamics



Effective dynamics



$$H = H_0 + \left(GH_1 + \frac{G^2}{c^2} H_2 + \frac{G^3}{c^4} H_3 + \frac{G^4}{c^6} H_4 \right) \left(1 + \frac{1}{c^2} + \dots \right)$$

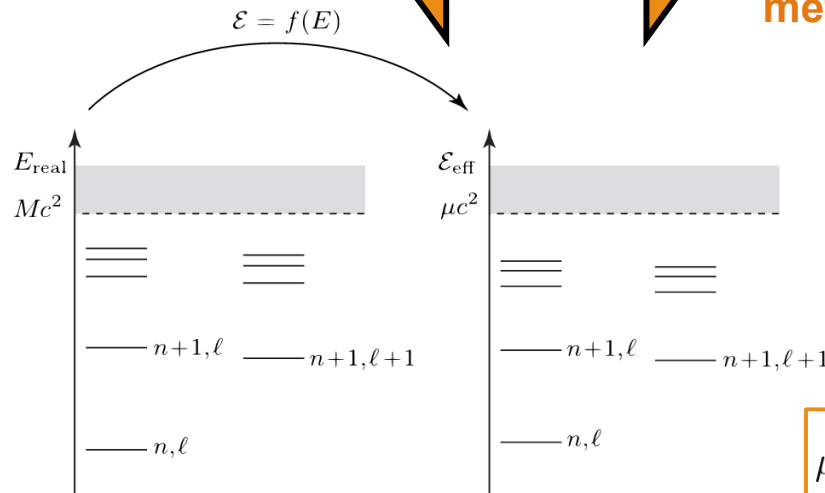
Effective metric

$$ds_{\text{eff}}^2 = -A(r)dt^2 + B(r)dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

Two-body/EOB “correspondence”: think quantum-mechanically (Wheeler)

Real 2-body system (m_1, m_2)
(in the c.o.m. frame)

an effective particle of
mass μ in some effective
metric $g_{\mu\nu}^{\text{eff}}(M)$



$$\mu^2 + g_{\text{eff}}^{\mu\nu} \frac{\partial S_{\text{eff}}}{\partial x^\mu} \frac{\partial S_{\text{eff}}}{\partial x^\nu} + \mathcal{O}(p^4) = 0$$

Figure 1: Sketch of the correspondence between the quantized energy levels of the real and effective conservative dynamics. n denotes the ‘principal quantum

Sommerfeld “Old
Quantum Mechanics”:

$$J = l\hbar = \frac{1}{2\pi} \oint p_\varphi d\varphi$$

$$N = n\hbar = I_r + J$$

$$I_r = \frac{1}{2\pi} \oint p_r dr$$

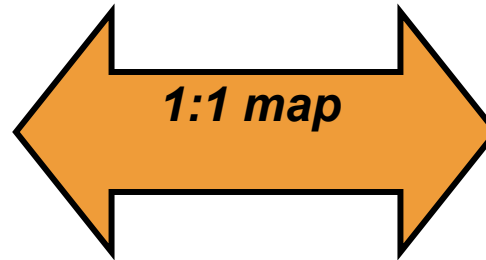
$H^{\text{classical}}(q,p)$

$H^{\text{classical}}(I_a)$

$E^{\text{quantum}}(I_a = n_a h) = f^{-1} [\mathcal{E}_{\text{eff}}^{\text{quantum}}(I_a^{\text{eff}} = n_a h)]$

The EOB energy map

Real 2-body system (m_1, m_2)
(in the c.o.m. frame)



an effective particle of
Mass $\mu = m_1 m_2 / (m_1 + m_2)$ in
some effective
metric $g_{\mu\nu}^{\text{eff}}(M)$

Simple energy map

$$\mathcal{E}_{\text{eff}} = \frac{s - m_1^2 - m_2^2}{2M}$$

$$s = E_{\text{real}}^2$$

$$H_{\text{EOB}} = M \sqrt{1 + 2\nu (\hat{H}_{\text{eff}} - 1)}$$

$$M = m_1 + m_2$$
$$\nu = m_1 m_2 / (m_1 + m_2)^2$$

Explicit form of the 3PN EOB effective Hamiltonian

The effective metric $g_{\mu\nu}^{\text{eff}}(M)$ at 3PN

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

where the coefficients are a ν -dependent “deformation” of the Schwarzschild ones:

$$A_{3\text{PN}}(R) = 1 - 2u + 2\nu u^3 + a_4\nu u^4$$

$$a_4 = \frac{94}{3} - \frac{41}{32}\pi^2 \simeq 18.6879027$$

$$(A(R)B(R))_{3\text{PN}} = 1 - 6\nu u^2 + 2(3\nu - 26)\nu u^3$$

$$u = GM/(c^2r)$$

Simple effective Hamiltonian

$$\hat{H}_{\text{eff}} \equiv \sqrt{p_{r_*}^2 + A \left(1 + \frac{p_\varphi^2}{r^2} + z_3 \frac{p_{r_*}^4}{r^2} \right)}.$$

$$p_{r_*} = \left(\frac{A}{B} \right)^{1/2} p_r$$

crucial EOB “radial potential” $A(r)$

2-body Taylor-expanded 3PN Hamiltonian [JS98, DJS00,01]

$$H_N(\mathbf{x}_a, \mathbf{p}_a) = \sum_a \frac{\mathbf{p}_a^2}{2m_a} - \frac{1}{2} \sum_a \sum_{b \neq a} \frac{Gm_a m_b}{r_{ab}}$$

$$H_{1PN}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8} \frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8} \frac{Gm_1 m_2}{r_{12}} \left[-12 \frac{\mathbf{p}_1^2}{m_1^2} + 14 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + 2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \right] + \frac{1}{4} \frac{Gm_1 m_2}{r_{12}} \frac{G(m_1 + m_2)}{r_{12}} + (1 \leftrightarrow 2),$$

$$H_{2PN}(\mathbf{x}_a, \mathbf{p}_a) = \frac{1}{16} \frac{(\mathbf{p}_1^2)^3}{m_1^4} + \frac{1}{8} \frac{Gm_1 m_2}{r_{12}} \left[5 \frac{(\mathbf{p}_1^2)^2}{m_1^3} - \frac{11}{2} \frac{\mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} - \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} + 5 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right. \\ \left. - 6 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} - \frac{3}{2} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right] \\ + \frac{1}{4} \frac{G^2 m_1 m_2}{r_{12}^2} \left[m_2 \left(10 \frac{\mathbf{p}_1^2}{m_1^2} + 19 \frac{\mathbf{p}_2^2}{m_2^2} \right) - \frac{1}{2} (m_1 + m_2) \frac{27 (\mathbf{p}_1 \cdot \mathbf{p}_2) + 6 (\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \right] \\ - \frac{1}{8} \frac{Gm_1 m_2}{r_{12}} \frac{G^2 (m_1^2 + 5m_1 m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2).$$

1PN

2PN

$$H_{3PN}^{\text{reg}}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{5}{128} \frac{(\mathbf{p}_1^2)^4}{m_1^5} + \frac{1}{32} \frac{Gm_1 m_2}{r_{12}} \left[-14 \frac{(\mathbf{p}_1^2)^3}{m_1^4} + 4 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 4 \mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^3 m_2^2} \mathbf{p}_1^2 + \frac{(\mathbf{p}_1^2 \mathbf{p}_2^2 - 2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2^3} \right. \\ \left. - 10 \frac{(\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 + \mathbf{p}_2^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2) \mathbf{p}_1^2}{m_1^4 m_2^3} + 24 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^4 m_2^3} + 2 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} \right. \\ \left. + \frac{7 \mathbf{p}_1^2 \mathbf{p}_2^2 - 10 (\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} (\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2) + 6 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^4 m_2^3} \right. \\ \left. + 15 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} - 18 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^4 m_2^3} + 5 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3} \right] \\ + \frac{G^2 m_1 m_2}{r_{12}^2} \left[\frac{1}{16} (m_1 - 27m_2) \frac{(\mathbf{p}_1^2)^2}{m_1^3} - \frac{115}{16} m_1 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2} + \frac{1}{48} m_2 \frac{25 (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 371 \mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} \right. \\ \left. + \frac{17 \mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{16 m_1^3} - \frac{1}{8} m_1 \frac{(15 \mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2) + 11 (\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2) + \frac{5}{12} (\mathbf{n}_{12} \cdot \mathbf{p}_1)^4)}{m_1^3 m_2} \right. \\ \left. - \frac{3}{2} m_1 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2} + \frac{125}{12} m_2 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} + \frac{10}{3} m_2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right. \\ \left. - \frac{1}{48} (220m_1 + 193m_2) \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^2} \right] + \frac{G^3 m_1 m_2}{r_{12}^3} \left[-\frac{1}{48} \left(466 m_1^2 + \left(473 - \frac{3}{4} \pi^2 \right) m_1 m_2 + 150 m_2^2 \right) \frac{\mathbf{p}_1^2}{m_1^3} \right. \\ \left. + \frac{1}{16} \left(77(m_1^2 + m_2^2) + \left(143 - \frac{1}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \frac{1}{16} \left(61 m_1^2 - \left(43 + \frac{3}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^3} \right. \\ \left. + \frac{1}{16} \left(21(m_1^2 + m_2^2) + \left(119 + \frac{3}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \right] \\ + \frac{1}{8} \frac{G^4 m_1 m_2^3}{r_{12}^4} \left[\left(\frac{227}{3} - \frac{21}{4} \pi^2 \right) m_1 + m_2 \right] + (1 \leftrightarrow 2). \quad (12)$$

3PN

NR-completed resummed 5PN EOB radial A potential

4PN analytically complete + 5 PN logarithmic term in the $A(u, \nu)$ function,
With $u = GM/R$ and $\nu = m_1 m_2 / (m_1 + m_2)^2$

[Damour 09, Blanchet et al 10, Barack-Damour-Sago 10, Le Tiec et al 11,
Barausse et al 11, Akcay et al 12, Bini-Damour 13,
Damour-Jaranowski-Schäfer 14, Nagar-Damour-Reisswig-Pollney 15]

$$\begin{aligned} A(u; \nu, a_6^c) &= P_5^1 \left[1 - 2u + 2\nu u^3 + \nu \left(\frac{94}{3} - \frac{41}{32} \pi^2 \right) u^4 \right. \\ &+ \nu \left[-\frac{4237}{60} + \frac{2275}{512} \pi^2 + \left(-\frac{221}{6} + \frac{41}{32} \pi^2 \right) \nu + \frac{64}{5} \ln(16e^{2\gamma} u) \right] u^5 \\ &\left. + \nu \left[a_6^c(\nu) - \left(\frac{7004}{105} + \frac{144}{5} \nu \right) \ln u \right] u^6 \right] \end{aligned}$$

$$a_6^{c \text{ NR-tuned}}(\nu) = 81.38 - 1330.6 \nu + 3097.3 \nu^2$$

2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014]

$$\begin{aligned}
 c^8 H_{4\text{PN}}^{\text{local}}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{7(\mathbf{p}_1^2)^5}{256m_1^9} + \frac{Gm_1m_2}{r_{12}} H_{48}(\mathbf{x}_a, \mathbf{p}_a) + \frac{G^2m_1m_2}{r_{12}^2} m_1 H_{46}(\mathbf{x}_a, \mathbf{p}_a) \\
 &+ \frac{G^3m_1m_2}{r_{12}^3} (m_1^2 H_{441}(\mathbf{x}_a, \mathbf{p}_a) + m_1m_2 H_{442}(\mathbf{x}_a, \mathbf{p}_a)) \\
 &+ \frac{G^4m_1m_2}{r_{12}^4} (m_1^3 H_{421}(\mathbf{x}_a, \mathbf{p}_a) + m_1^2m_2 H_{422}(\mathbf{x}_a, \mathbf{p}_a)) \\
 &+ \frac{G^5m_1m_2}{r_{12}^5} H_{40}(\mathbf{x}_a, \mathbf{p}_a) + (1 \leftrightarrow 2), \tag{A3}
 \end{aligned}$$

$$\begin{aligned}
 H_{48}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{45(\mathbf{p}_1^2)^4}{128m_1^8} - \frac{9(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64m_1^6m_2^2} + \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64m_1^6m_2^2} - \frac{9(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{16m_1^6m_2^2} \\
 &- \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{32m_1^6m_2^2} + \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{35(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{256m_1^6m_2^2} \\
 &+ \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2}{128m_1^6m_2^2} + \frac{33(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256m_1^6m_2^2} - \frac{85(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^6m_2^2} \\
 &- \frac{45(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{128m_1^6m_2^2} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^6m_2^2} + \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^6m_2^2} \\
 &+ \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^6m_2^2} - \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{64m_1^6m_2^2} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{64m_1^6m_2^2} + \frac{55(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_2^2}{256m_1^6m_2^2} \\
 &- \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128m_1^6m_2^2} - \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256m_1^6m_2^2} - \frac{23(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{256m_1^6m_2^2} \\
 &+ \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{128m_1^6m_2^2} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{256m_1^6m_2^2} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{64m_1^6m_2^2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64m_1^6m_2^2} \\
 &- \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{4m_1^6m_2^2} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16m_1^6m_2^2} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{64m_1^6m_2^2} + \frac{21(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64m_1^6m_2^2} \\
 &- \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32m_1^6m_2^2} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{4m_1^6m_2^2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{16m_1^6m_2^2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{16m_1^6m_2^2} \\
 &- \frac{\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{32m_1^6m_2^2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64m_1^6m_2^2} - \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32m_1^6m_2^2} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128m_1^6m_2^2}, \tag{A4a}
 \end{aligned}$$

$$\begin{aligned}
 H_{46}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{369(\mathbf{n}_{12} \cdot \mathbf{p}_1)^6}{160m_1^6} - \frac{889(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4\mathbf{p}_1^2}{192m_1^6} + \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64m_1^6} - \frac{549(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{128m_1^6m_2} \\
 &+ \frac{67(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{16m_1^6m_2} - \frac{167(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2}{128m_1^6m_2} + \frac{1547(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^6m_2} - \frac{851(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{128m_1^6m_2} \\
 &+ \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^6m_2} + \frac{3263(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{1280m_1^6m_2^2} + \frac{1067(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{480m_1^6m_2^2} - \frac{4567(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840m_1^6m_2^2} \\
 &- \frac{3571(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{320m_1^6m_2^2} + \frac{3073(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{480m_1^6m_2^2} + \frac{4349(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{1280m_1^6m_2^2} \\
 &- \frac{3461\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{3840m_1^6m_2^2} + \frac{1673(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4\mathbf{p}_2^2}{1920m_1^6m_2^2} - \frac{1999(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840m_1^6m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840m_1^6m_2^2} - \frac{13(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{8m_1^6m_2^2} \\
 &+ \frac{191(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2}{192m_1^6m_2^2} - \frac{19(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^6m_2^2} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^6m_2^2} \\
 &+ \frac{11(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{192m_1^6m_2^2} + \frac{77(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{96m_1^6m_2^2} + \frac{233(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_2^2}{96m_1^6m_2^2} - \frac{47(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^6m_2^2} \\
 &+ \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{384m_1^6m_2^2} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{384m_1^6m_2^2} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4}{4m_1^6m_2^2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4\mathbf{p}_1^2}{4m_1^6m_2^2} \\
 &- \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3(\mathbf{p}_1 \cdot \mathbf{p}_2)}{2m_1^6m_2^2} - \frac{21(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16m_1^6m_2^2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{6m_1^6m_2^2} + \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48m_1^6m_2^2} \\
 &- \frac{133(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{24m_1^6m_2^2} - \frac{77(\mathbf{p}_1 \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{96m_1^6m_2^2} + \frac{197(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96m_1^6m_2^2} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48m_1^6m_2^2} + \frac{13(\mathbf{p}_2^2)^3}{8m_2^6}, \tag{A4b}
 \end{aligned}$$

$$\begin{aligned}
 H_{441}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{5027(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{384m_1^4} - \frac{22993(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2}{960m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152m_1^4} - \frac{3191(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{640m_1^3m_2} \\
 &+ \frac{28561(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{1920m_1^3m_2} + \frac{8777(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^3m_2} + \frac{752969\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{28800m_1^3m_2} \\
 &- \frac{16481(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{960m_1^3m_2^2} + \frac{94433(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{4800m_1^3m_2^2} - \frac{103957(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{2400m_1^3m_2^2} \\
 &+ \frac{791(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{400m_1^3m_2^2} + \frac{26627(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_2^2}{1600m_1^3m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800m_1^3m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32m_2^3}, \tag{A4c}
 \end{aligned}$$

$$\begin{aligned}
 H_{442}(\mathbf{x}_a, \mathbf{p}_a) &= \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200} \right) \frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{m_1^4} \\
 &+ \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600} \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^3m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600} \right) \frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^3m_2^2} \\
 &+ \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^3m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3m_2^2} \\
 &- \left(\frac{30383}{960} + \frac{36405\pi^2}{16384} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384} \right) \frac{\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3m_2^2} \\
 &+ \left(\frac{2369}{60} + \frac{35655\pi^2}{16384} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3m_2} + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{m_1^3m_2} \\
 &+ \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3m_2}, \tag{A4d}
 \end{aligned}$$

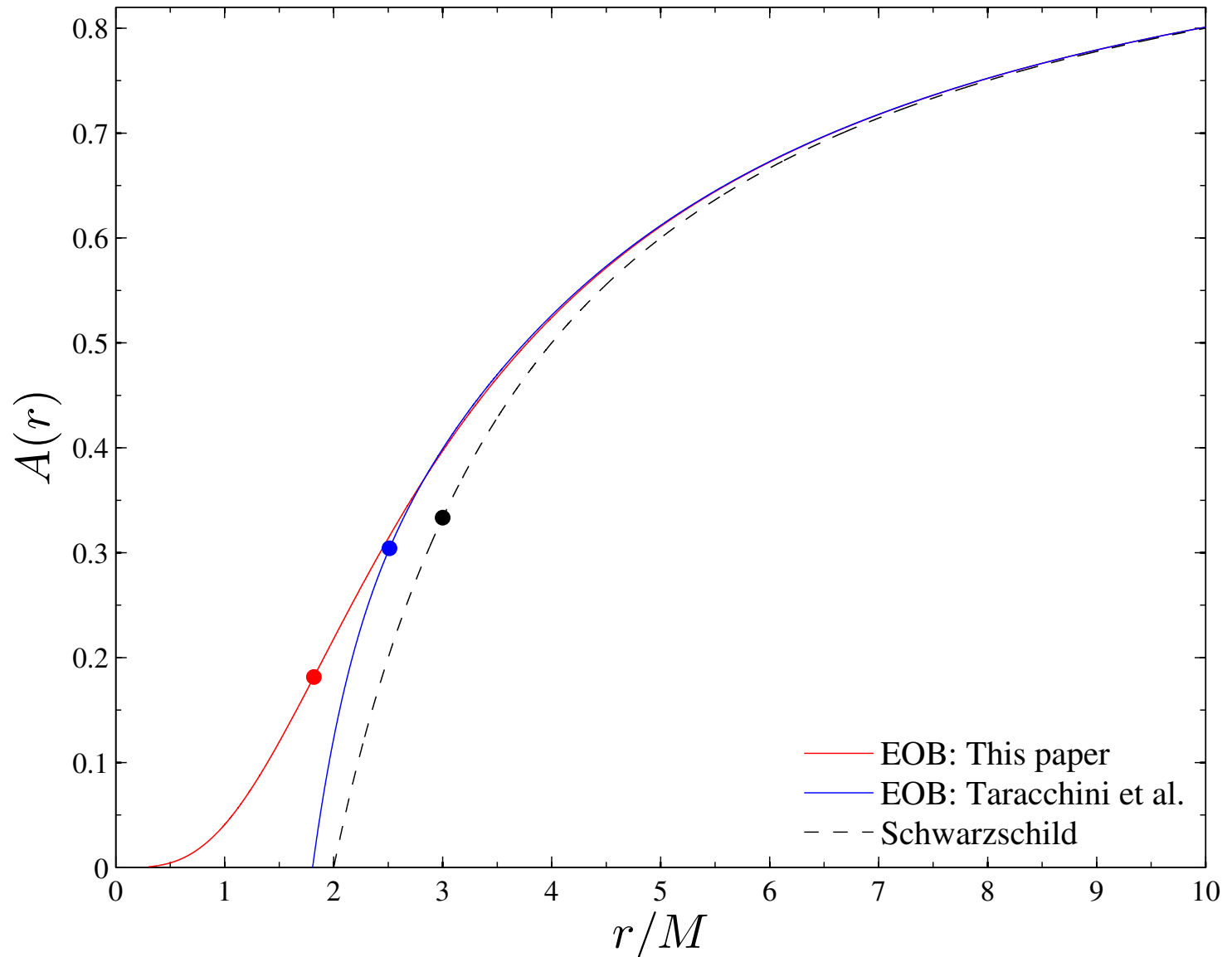
$$H_{421}(\mathbf{x}_a, \mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800m_1^2} - \frac{91(\mathbf{p}_1 \cdot \mathbf{p}_2)}{8m_1m_2} + \frac{105\mathbf{p}_2^2}{32m_2^2} - \frac{9841(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{1600m_1^2} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{2m_1m_2}, \tag{A4e}$$

$$\begin{aligned}
 H_{422}(\mathbf{x}_a, \mathbf{p}_a) &= \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152} \right) \frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600} \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192} \right) \frac{\mathbf{p}_2^2}{m_2^2} \\
 &+ \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} + \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1m_2} \\
 &+ \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_2^2}, \tag{A4f}
 \end{aligned}$$

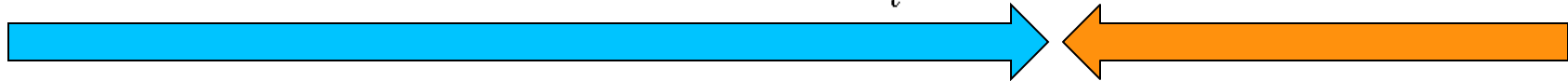
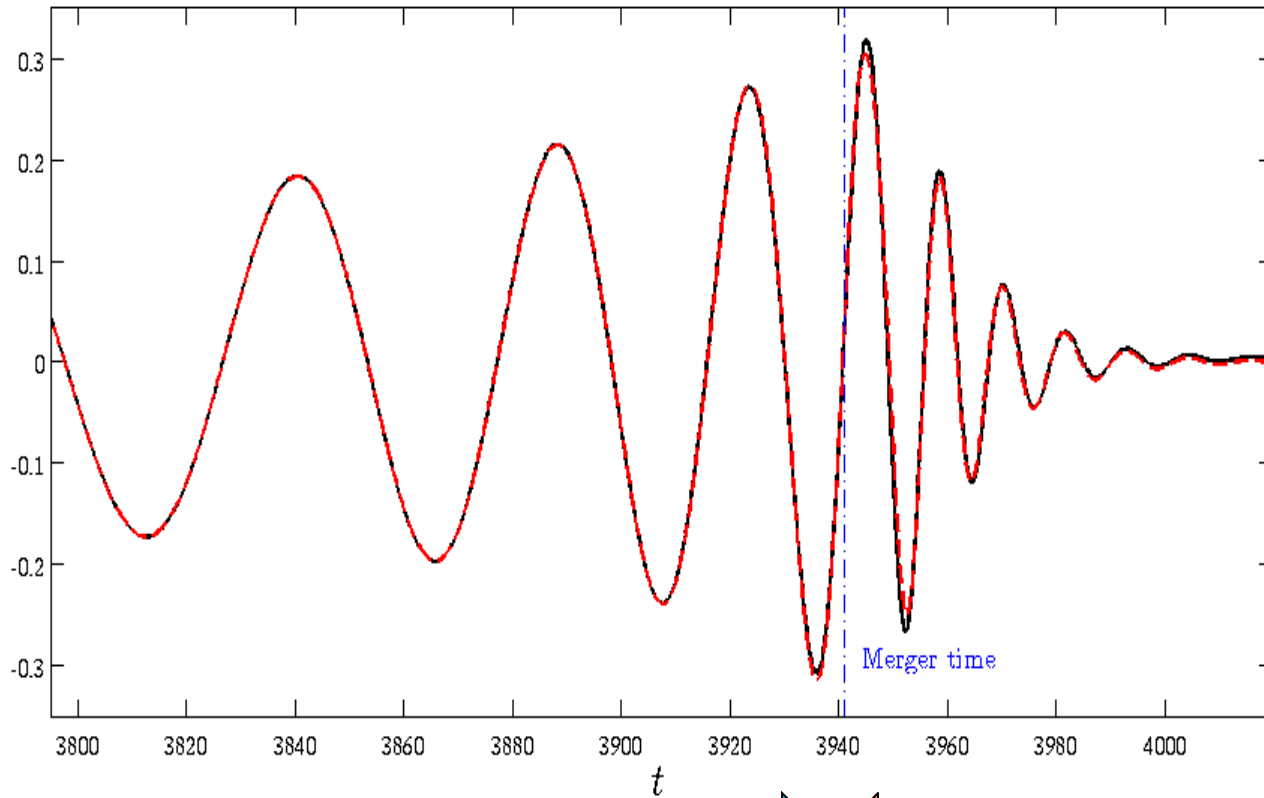
$$H_{40}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400} \right) m_1^3m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200} \right) m_1^2m_2^2. \tag{A4g}$$

$$\begin{aligned}
 H_{4\text{PN}}^{\text{nonloc}}(t) &= -\frac{1}{5} \frac{G^2M}{c^8} I_{ij}^{(3)}(t) \\
 &\times \text{Pf}_{2r_{12}/c} \int_{-\infty}^{+\infty} \frac{dv}{|v|} I_{ij}^{(3)}(t+v),
 \end{aligned}$$

Comparing the EOB A potentials of Nagar et al 15 and Taracchini et al 14 to $A^{\text{Schw}} = 1 - 2M/r$

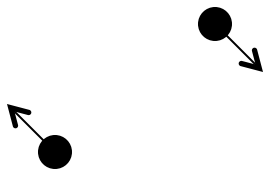


Binary BH coalescence: Analytical Relativity vs NR[CC]



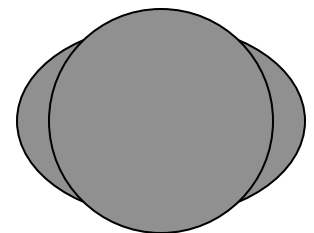
Inspiral + « plunge »

Ringdown

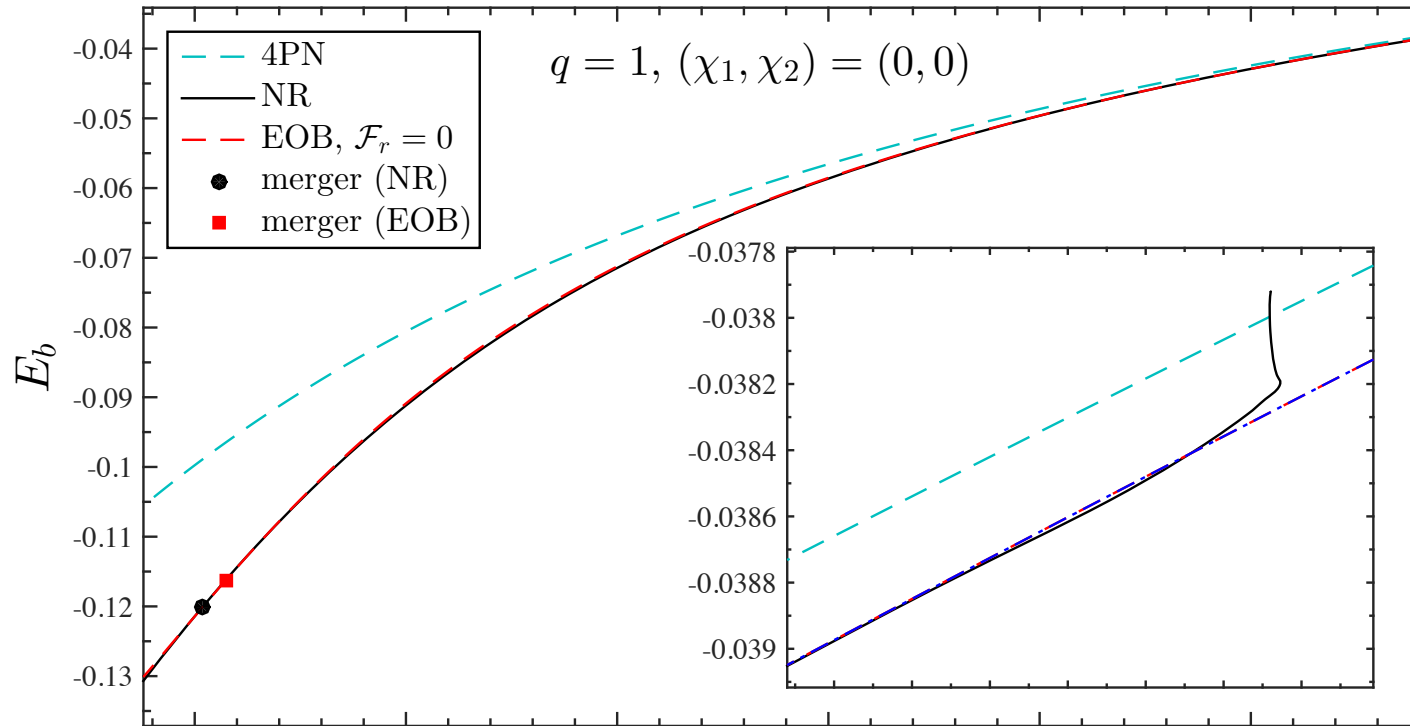


Two orbiting point-masses:
Resummed dynamics

Ringling BH

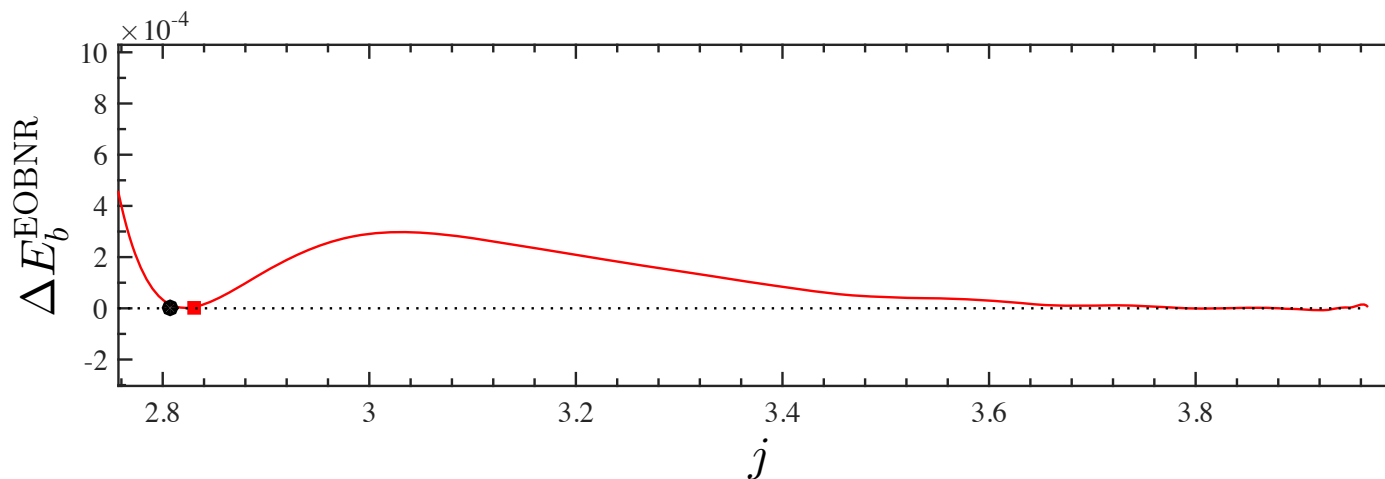


Energy and Angular Momentum in Spinning and Nonspinning Black-Hole Binaries : E(J)



Damour, Nagar,
Pollney, Reisswig 12

Nagar, Damour,
Reisswig, Pollney 15



$q(x)$ and periastron advance in (comparable-mass) Black Hole binaries (Le Tiec et al. 2011)

$$K(x) = \Omega_\phi / \Omega_r; \quad x = (M \Omega_\phi)^{2/3}$$

$$K_{\text{EOB}} = \sqrt{\frac{A'_p(u)}{D(u)\Delta(u)}}$$

$$K_{\text{GSF}}^v = \frac{1}{\sqrt{1-6x}} \left[1 - \frac{v}{2} \frac{\rho(x)}{1-6x} + \mathcal{O}(v^2) \right],$$

where $A'_p = dA_p/du$, and $\Delta = A_p A'_p + 2u(A'_p)^2 - u A_p A''_p$

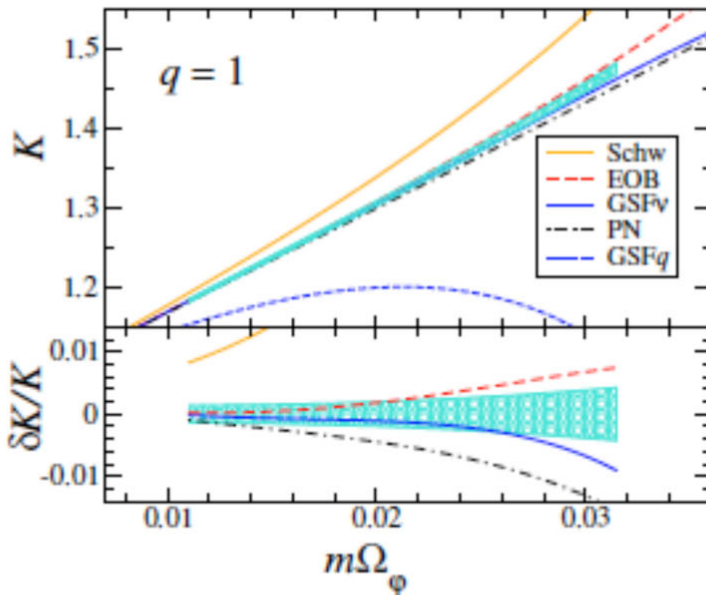


FIG. 1: The periastron advance K of an equal mass black hole binary, in the limit of zero eccentricity, as a function of the orbital frequency Ω_ϕ of the circular motion. The NR results are indicated by the cyan-shaded region. The PN and EOB results are valid at 3PN order. The lower panel shows the relative difference $\delta K/K \equiv (K - K_{\text{NR}})/K_{\text{NR}}$.

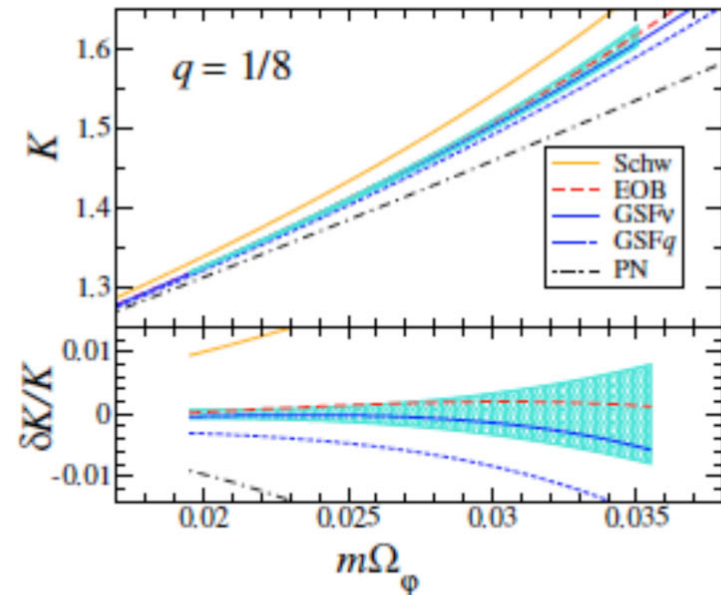
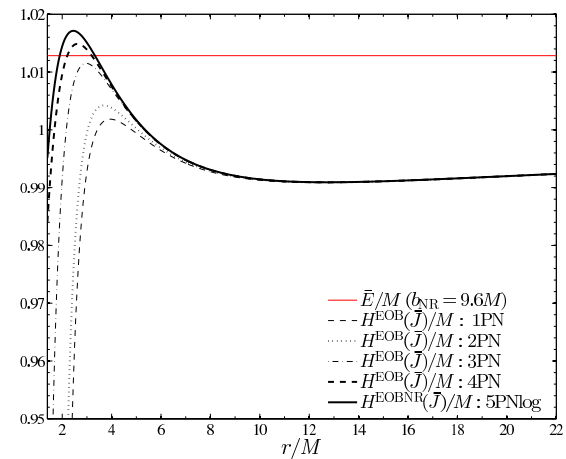
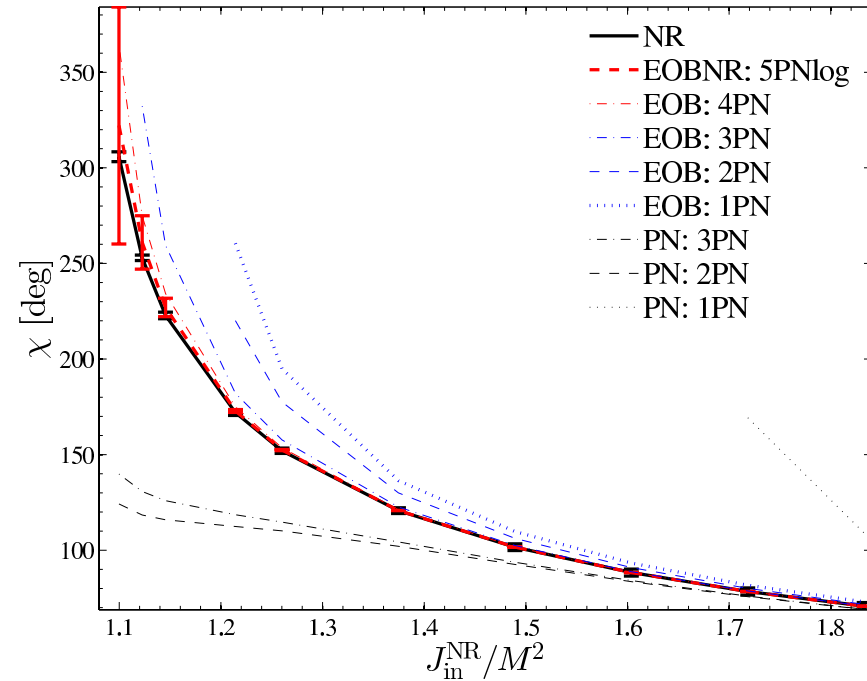
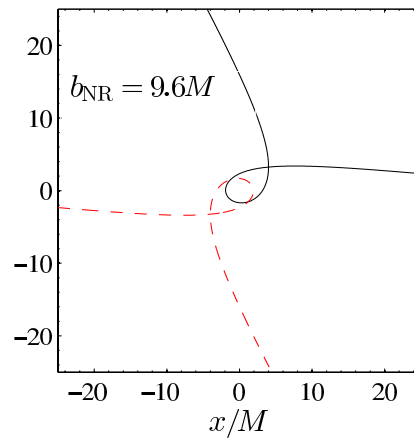
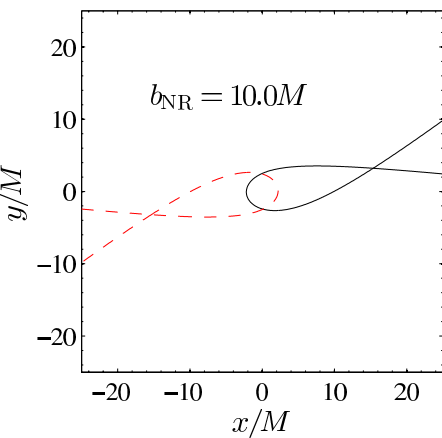
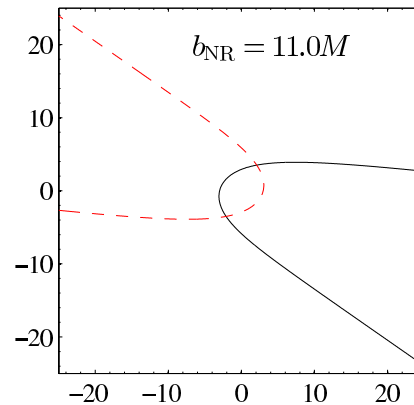
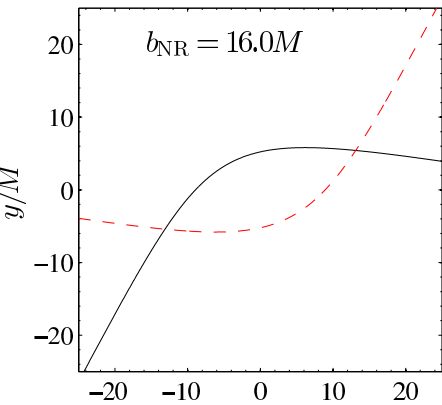


FIG. 2: Same as in Fig. 1, but for a mass ratio $q = 1/8$. Note that for an orbital frequency $m\Omega_\phi \sim 0.03$, corresponding to a separation $r \sim 10m$, the periastron advance reaches half an orbit per radial period.

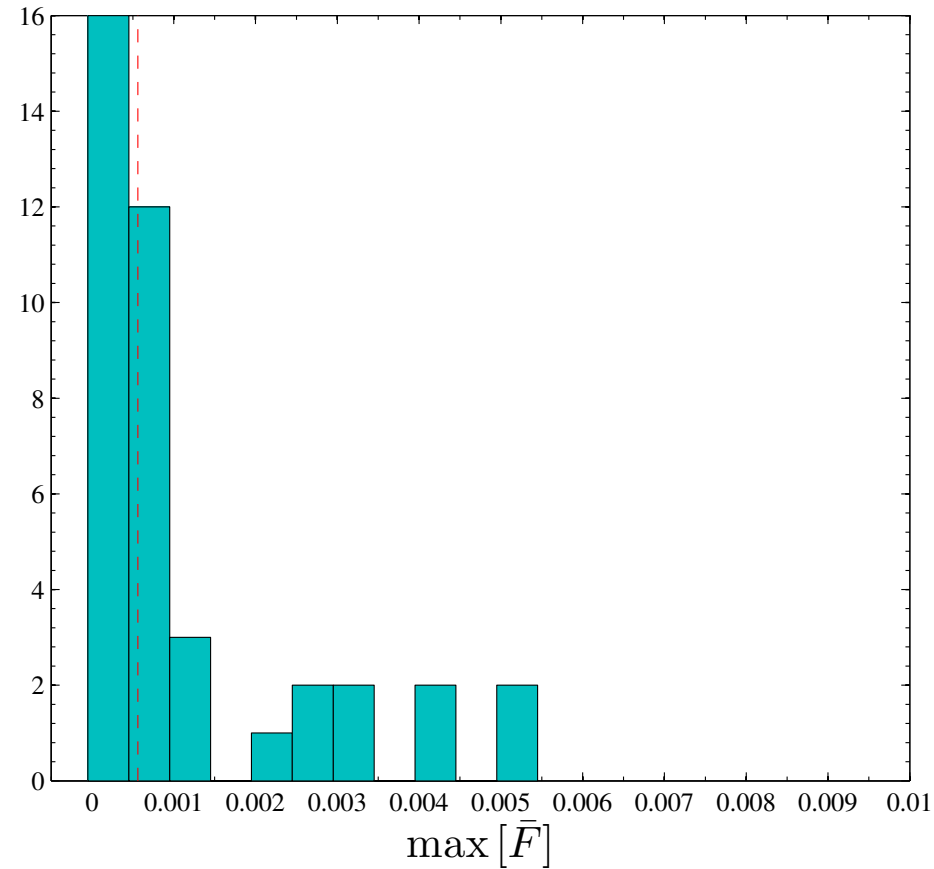
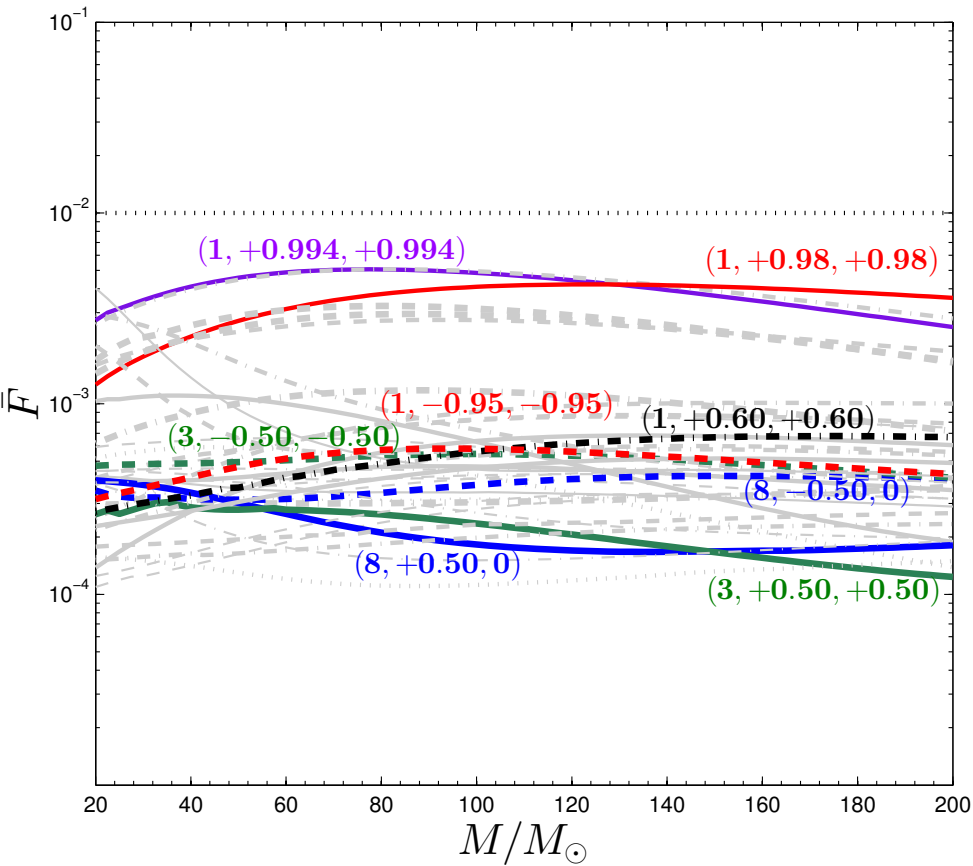
Since $\rho(x) > 0$ for all stable circular orbits, the $\mathcal{O}(q)$ GSF decreases the rate of precession. Note that the fastest disc

Strong-Field Scattering of Two Black Holes

(Damour, Guercilena, Hinder, Hopper, Nagar, Rezzolla 2014)

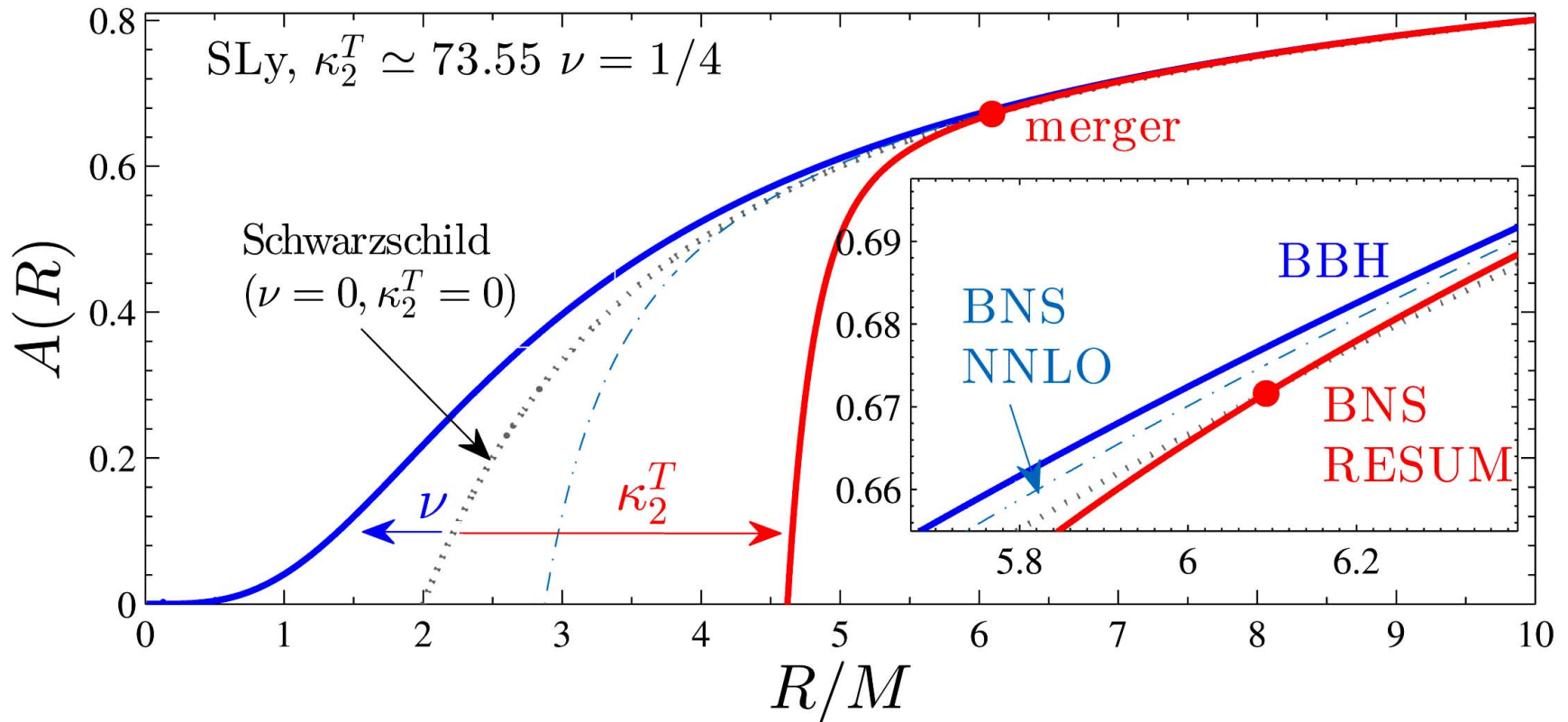


EOB/NR unfaithfulness = $1 - F$ for spinning BBH



Binary neutron stars: Tidal EOB [NR] potential

Bernuzzi, Nagar, Dietrich & Damour 2015 [based on Bini-Damour 2014]



Conclusions on the “Problem of Motion”

- Poincaré : “Il n’y a pas de problèmes résolus et d’autres qui ne le sont pas; Il y a seulement des problèmes *plus ou moins* résolus.”
- **Experimentally**, gravitational wave astronomy is about to start. The ground-based network of detectors (LIGO/Virgo/GEO/...) is being updated (ten-fold gain in sensitivity in a few years), and extended (KAGRA, LIGO-India).
- One of the **prime GW sources** that one expects to detect is the GW signal emitted by coalescing binary systems (BH, NS)
- The study of the **motion and radiation of binary systems** recently made a lot of progress, in several different directions : **PN, GSF, EOB, EFT, NR**. There is a useful **synergy** between the various methods (e.g. recent analytical modelling of BNS tidal interaction)
- There exists a **complementarity** between Numerical Relativity and **Analytical Relativity**, especially when using the particular **resummation** of perturbative results defined by the **Effective One Body** formalism. The **NR-tuned EOB** formalism is likely to be essential for computing ⁴¹ the many thousands of accurate GW templates needed for LIGO/Virgo/...